Math 511 $\qquad x_1y_1 + x_2y_2 + \cdots + x_ny_n$

$\mathbb{R}^n$ we use $\boxed{\mathbf{x}^T\mathbf{y}}$ to understand ..

length : $\|\mathbf{x}\| = (\mathbf{x}^T\mathbf{x})^{1/2}$

orthogonal : $\mathbf{x}^T\mathbf{y} = 0$

angle : $\cos\theta = \dfrac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|}$ $\qquad \theta \in [0,\pi]$

projection: $\mathbb{P}$ is $\mathbf{x}$ projected onto $\mathbf{y}$

$$\alpha = \|\mathbb{P}\| = \frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{y}\|}$$

$$\mathbb{P} = \alpha\frac{\mathbf{y}}{\|\mathbf{y}\|} = \frac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\,\mathbf{y}$$

5.4 given V, a vector space, with an operator

objects with $\mathbf{x}+\mathbf{y}$, $\alpha\mathbf{x}$ defined

that assigns to $\mathbf{x},\mathbf{y} \in V$ a real number

Notation: ① $\langle \mathbf{x}, \mathbf{y} \rangle \in \mathbb{R}$

② inner product

Such that Ⅰ $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$

and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ iff $\mathbf{x} = \mathbf{0}$

(II) $\langle \mathbb{X}, \mathbb{Y} \rangle = \langle \mathbb{Y}, \mathbb{X} \rangle$

(III) $\langle \alpha \mathbb{X} + \beta \mathbb{Y}, \mathbb{Z} \rangle = \alpha \langle \mathbb{X}, \mathbb{Z} \rangle + \beta \langle \mathbb{Y}, \mathbb{Z} \rangle$

Def V, a vector space, with $\langle \mathbb{X}, \mathbb{Y} \rangle$ defined is called an $\boxed{\text{inner product space}}$

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ex (1) $\mathbb{R}^n$ with $\langle \mathbb{X}, \mathbb{Y} \rangle = \mathbb{X}^T \mathbb{Y}$ (scalar prod)

$\langle \mathbb{X}, \mathbb{Y} \rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$

check (I) is $\langle \mathbb{X}, \mathbb{X} \rangle \geq 0$ and $= 0$ only $\mathbb{X} = \mathbb{0}$?

$\langle \mathbb{X}, \mathbb{X} \rangle = x_1^2 + x_2^2 + \dots + x_n^2$ ✓

(II) $\langle \mathbb{X}, \mathbb{Y} \rangle = \langle \mathbb{Y}, \mathbb{X} \rangle$ ✓

(III) show $\langle \alpha \mathbb{X} + \beta \mathbb{Y}, \mathbb{Z} \rangle = \alpha \langle \mathbb{X}, \mathbb{Z} \rangle + \beta \langle \mathbb{Y}, \mathbb{Z} \rangle$ ✓

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(2) $\mathbb{R}^n$ with $\langle \mathbb{X}, \mathbb{Y} \rangle = \boxed{w_1 x_1 y_1 + w_2 x_2 y_2 + \dots + w_n x_n y_n}$ weighted scalar product

with $w_i > 0$ called weights

Note: we are studying $\langle \mathbb{X}, \mathbb{Y} \rangle$ b/c the scalar prod. for $\mathbb{R}^n$ was useful!

→ most likely any $\langle X, Y \rangle$ for a V based in a way on $X^T Y$ scalar prod will probably be useful.

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(3) $\mathbb{R}^{m \times n}$ define $\langle A, B \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij}$

this inner product does satisfy (I), (II) and (III)

(4) $\mathbb{R}^{m \times n}$ define $\langle A, B \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} a_{ij} b_{ij}$

$w_{ij} > 0$ weights

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(5) $C[a,b]$ define $\langle f, g \rangle = \int_a^b f(x) g(x)\, dx$

this does satisfy (I), (II), and (III)

(6) $C[a,b]$ define $\langle f, g \rangle = \int_a^b w(x) f(x) g(x)\, dx$

$w(x) > 0$ is the weight function

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(7) P_n → pick $x_1, x_2, \ldots, x_n$ points

$\langle p, q \rangle = p(x_1) q(x_1) + p(x_2) q(x_2) + \cdots + p(x_n) q(x_n)$

(3) $P_n \to$ pick $x_1, \dots, x_n$ points

$$\langle p, q \rangle = w(x_1)p(x_1)q(x_1) + \dots + w(x_n)\, p(x_n)\, q(x_n)$$

$\boxed{w(x) > 0}$ weight function.

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given V, vector space, with $\langle x, y \rangle$ inner product

to get an inner product space.

Q's "length", "angle", "orthogonal", "projection"?

(1) "length" $\|x\| = (\langle x, x \rangle)^{1/2}$

ex $\|1 + x\|$ for $C[0,1]$ with

$$\langle f, g \rangle = \int_0^1 fg\,dx$$

$$\|1 + x\| = \left(\int_0^1 (1+x)^2\, dx \right)^{1/2}$$

$$= \left(\int_0^1 1 + 2x + x^2\, dx \right)^{1/2} = \left(1 + 1 + \tfrac{1}{3}\right)^{1/2}$$

$$= \sqrt{7/3}$$

(2) "orthogonal" if $\langle x, y \rangle = 0$

ex is 1 and x orthogonal on $C[-2,2]$

with our normal non-weighted inner prod.

(i) $\langle 1, x \rangle \overset{?}{=} 0$

$$\langle 1, x \rangle = \int_{-2}^{2} (1)(x)\,dx = \int_{-2}^{2} x\,dx = 0$$

so $1 \perp x$

Thm Pythagorean law

$$\|x + y\|^2 = \langle x+y, x+y \rangle$$

$$= \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle$$

if $x \perp y \to \langle x, y \rangle\ 0$

$$\to \|x + y\|^2 = \|x\|^2 + \|y\|^2$$

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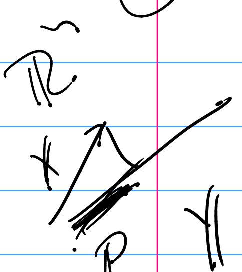
(3) "projections" P is the projection of x onto y



$$\alpha = \frac{\langle x, y \rangle}{\|y\|}$$

$$P = \alpha \frac{y}{\|y\|} = \frac{\langle x, y \rangle}{\langle y, y \rangle} y$$

ex. $\mathbb{R}^{2\times 2}$ $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

project A onto B using the non-weighted inner prod.

$$\alpha = \frac{\langle A, B\rangle}{\| B \|} = \frac{\langle A, B\rangle}{(\langle B, B\rangle)^{1/2}} = \frac{1}{\sqrt{7}}$$

$$P = \frac{\langle A, B\rangle}{\langle B, B\rangle} B = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2/7 & -1/7 \\ -1/7 & 1/7 \end{bmatrix}}$$

Thm Cauchy-Schwarz $\quad |\langle X, Y\rangle| \leq \|X\|\,\|Y\|$

(i) $\rightarrow$ (angles) $\quad \cos\theta = \dfrac{\langle X, Y\rangle}{\|X\|\,\|Y\|} \qquad \theta \in [0, \pi]$

Note: all the ideas of "length", "orthogonal", etc apply to any inner product space.

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so we can...

(ex) project $f(x) = \cos x$ onto $g(x) = 1 + x + x^2$ for $C[-\pi/2, \pi/2]$ with $\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} fg\,dx$

Inner Prod Space

$$p(x) = \frac{\langle f, g\rangle}{\langle g, g\rangle} \cdot g$$

$$p(x) = \frac{\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} (\cos x)(1+x+x^2)\,dx}{\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} (1+x+x^2)^2\,dx}\,(1+x+x^2)$$

Note: Inner Product space : Vector Space + inner product

objects with ideas of length + angles

→ $\boxed{\text{Linear Normed Space}}$

Def: define a function $\|X\|$, norm of X, such that

I. $\|X\| \geq 0$ and $= 0$ iff $X = 0$

II. $\|\alpha X\| = |\alpha|\,\|X\|$

III. $\|X + Y\| \leq \|X\| + \|Y\|$

$\mathbb{R}^n$

p-norm $\|X\|_p = \left(|x_1|^p + |x_2|^p + \dots + |x_n|^p\right)^{1/p}$

$\|X\|_\infty = \max |x_i|$