Math 511

Convert Basis: given Vector Space, $\mathbb{Q}$

$\dim(\mathbb{Q}) = K$

$\rightarrow$ basis $B = [\mathbb{b}_1\ \mathbb{b}_2 \cdots \mathbb{b}_K]$

$\rightarrow$ basis $D = [\mathbb{d}_1\ \mathbb{d}_2 \cdots \mathbb{d}_K]$

(always have standard $E = [\mathbb{e}_1\ \mathbb{e}_2 \cdots \mathbb{e}_K]$)

Notation:

$\mathbb{x}$ or $\mathbb{x}_E$ coord. in standard basis

$\mathbb{x}_B$ coord in basis B

$\mathbb{x}_D$ coord in basis D

Change of Basis

$$\mathbb{x}_E = B\,\mathbb{x}_B$$

$$\mathbb{x}_B = B^{-1}\,\mathbb{x}_E$$

$$\mathbb{x}_D = D^{-1} B\,\mathbb{x}_B$$

$$\mathbb{x}_B = \underbrace{\boxed{B^{-1} D}}_{\text{matrix to convert from } D \text{ to } B \text{ basis}}\,\mathbb{x}_D$$

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ex $\mathbb{R}^2$ basis: $B = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 1 & -4 \end{bmatrix}$

Find matrix S which converts from D to B

$$\mathbb{x}_B = \underbrace{\boxed{B^{-1} D}}_{S}\,\mathbb{x}_D$$

$$S = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1 & -4 \end{bmatrix}$$

tech #1 ① Find $\begin{bmatrix}-2 & -1\\ 1 & 0\end{bmatrix}^{-1} \begin{pmatrix}7\\6\end{pmatrix}$

② $\begin{pmatrix}3\end{pmatrix}\begin{bmatrix}1 & 6\\ 1 & 4\end{bmatrix} = S$

tech #2

$[B \mid D]$
$\rightarrow [I \mid B^{-1}D]$

$$\left[\begin{array}{cc|cc} 2 & -1 & 1 & 0\\ 1 & 0 & 1 & -4\end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & -4\\ 2 & -1 & 1 & 0\end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & -4\\ 0 & -1 & -1 & 8\end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & -4\\ 0 & 1 & 1 & -8\end{array}\right]$$

$$S = \begin{bmatrix}1 & -4\\ 1 & -8\end{bmatrix} = B^{-1}D$$

$$x_B = \underset{B \to D}{S}\, x_D \quad \text{or} \quad (\underset{B \to D}{S})^{-1} x_B = x_D$$

4.2 given $L: \mathbb{R}^n \to \mathbb{R}^n$ linear transform

$\rightarrow$ Find a matrix, A_{standard}, that acts as L.

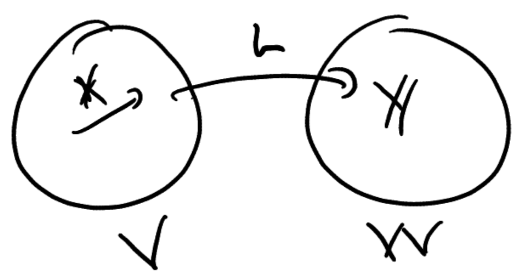
$$L(x) = A_{\text{stand}}\, x$$

$$A_{\text{stand}} = \left[L(e_1) \ \ L(e_2) \ \cdots \ L(e_n) \right]$$

Consider: Know: $L(x) = y$

given standard Basis

$$y = A_{\text{stand}}\, x$$
$$y = A x$$



But: Someone likes Basis $\mathcal{B} = [\mathbf{b}_1\ \mathbf{b}_2 \dots \mathbf{b}_n]$

and Basis $D = [\mathbf{d}_1\ \mathbf{d}_2 \dots \mathbf{d}_m]$

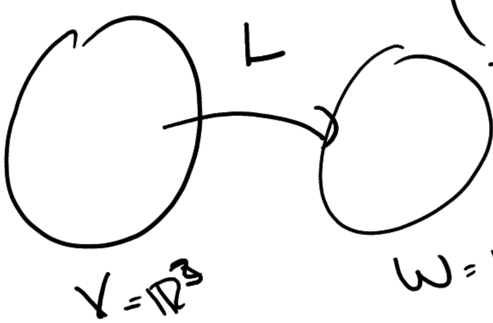
$$\mathbf{y}_D = \begin{bmatrix} ? \end{bmatrix} \mathbf{x}_B$$

find a matrix for L using Basis B to D

Have: L in standard is just matrix $\boxed{A}$

$$\mathbf{y}_D = \underbrace{\boxed{D^{-1} A\ B}}_{\substack{\shortparallel \\ T}} \mathbf{x}_B$$

so T is the transform matrix from basis B to D for L



L: is a linear transform.

ex: $L(\mathbf{x}) = \boxed{x_1}\,\mathbf{b}_1 + \boxed{(x_2+x_3)}\,\mathbf{b}_2$

$$L\begin{pmatrix}1\\0\\0\end{pmatrix} = 1\cdot\mathbf{b}_1 + (0+0)\,\mathbf{b}_2 = \begin{bmatrix}1\\0\end{bmatrix}_B$$

$$L\begin{pmatrix}0\\1\\0\end{pmatrix} = 0\,\mathbf{b}_1 + (1+0)\,\mathbf{b}_2 = \begin{bmatrix}0\\1\end{bmatrix}_B$$

$$L\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{bmatrix}0\\1\end{bmatrix}_B$$

$$\underset{B \leftarrow E}{M} = \left[L\begin{pmatrix}1\\0\\0\end{pmatrix}\ L\begin{pmatrix}0\\1\\0\end{pmatrix}\ L\begin{pmatrix}0\\0\\1\end{pmatrix} \right] = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\end{bmatrix}$$

$$\mathbf{y}_B = M\,\mathbf{x}_E$$

Notation: $\mathbf{x}$, $\mathbf{y}$, assume standard.

ex $L: \mathbb{R}^2 \to \mathbb{R}^3$

$$L(\mathbf{x}) = \begin{bmatrix} x_2 \\ x_1 + x_2 \\ x_1 - x_2 \end{bmatrix} \qquad L\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 5 \\ -3 \end{bmatrix}$$

$$L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} b \\ a+b \\ a-b \end{bmatrix}$$

is it a linear transform?

check
$$L(\alpha_1 \underset{\begin{bmatrix} a \\ b \end{bmatrix}}{\mathbf{v}_1} + \alpha_2 \underset{\begin{bmatrix} c \\ d \end{bmatrix}}{\mathbf{v}_2}) = \alpha_1 L(\underset{\begin{bmatrix} a \\ b \end{bmatrix}}{\mathbf{v}_1}) + \alpha_2 L(\underset{\begin{bmatrix} c \\ d \end{bmatrix}}{\mathbf{v}_2})$$

$$L\left(\begin{bmatrix} \alpha_1 a + \alpha_2 c \\ \alpha_1 b + \alpha_2 d \end{bmatrix}\right) \overset{?}{=} \alpha_1 L\begin{pmatrix} a \\ b \end{pmatrix} + \alpha_2 L\begin{pmatrix} c \\ d \end{pmatrix}$$

$$\begin{bmatrix} \alpha_1 b + \alpha_2 d \\ (\alpha_1 a + \alpha_2 c) + (\alpha_1 b + \alpha_2 d) \\ (\alpha_1 a + \alpha_2 c) - (\alpha_1 b + \alpha_2 d) \end{bmatrix} \overset{?}{=} \alpha_1 \begin{bmatrix} b \\ a+b \\ a-b \end{bmatrix} + \alpha_2 \begin{bmatrix} d \\ c+d \\ c-d \end{bmatrix}$$

Yes

Q1 what is A, the stand. matrix, for L?

$$\boxed{A = \left[L\begin{pmatrix} 1 \\ 0 \end{pmatrix} \; L\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]} = \boxed{\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}}$$

$$L(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{x}$$

ex
$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -3 \end{bmatrix}$$

Q2 $L = \mathbb{R}^2 \to \mathbb{R}^3$

Basis: $\underset{T}{\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}} \qquad \underset{R}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$

Find a matrix, S, represents L using bases T, R

$$\mathbb{Y}_R = \underbrace{R^{-1} A B}_{= S} \mathbb{X}_B$$

Happy: $L: \mathbb{R}^4 \to \mathbb{R}^3$

$$\text{Happy}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Matrix, ☺, that rep. Happy.

$$☺ = \left[L\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \; L\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \; L\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \; L\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right]$$

$$\boxed{☺} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Special Case: $L: V \to V$ want to use base S

$\to$ $A_{\text{standard}} = \left[L(e_1) \; L(e_2) \; .. \; L(e_n) \right]$

$\Rightarrow$ $\mathbb{Y}_S = \underbrace{\boxed{S^{-1} A S}}_{= T} \mathbb{X}_S$

Notice: L has two forms. (as a matrix)

① standard basis $\to$ A $\qquad \mathbb{Y} = A\mathbb{X}$

② our other basis: S $\to$ T $\qquad \mathbb{Y}_S = T \mathbb{X}_S$

Notation: $T = S^{-1} A S$

Def A and T are called $\boxed{\text{similar}}$

if there exists a non-singular matrix S

such that $T = S^{-1} A S$

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ex $L : V \to V$ and we have two interesting <u>bases</u>. B and D

Q1 Find std. matrix for L.

$$\boxed{A_{\text{std.}}} = \left[L(e_1) \ \ L(e_2) \cdots L(e_n) \right]$$

Q2 Find matrix for L using basis B (N)

$$\mathbf{y}_B = \underbrace{\boxed{B^{-1} A B}}_{N} \mathbf{x}_B$$

Q3 Find matrix, M, for L using basis D

$$\mathbf{y}_D = \underbrace{\boxed{D^{-1} A D}}_{M} \mathbf{x}_D$$

Q4 Find a matrix, S, for L going from B to D

$$\mathbf{y}_D = \underbrace{\boxed{D^{-1} A B}}_{S} \mathbf{x}_B$$

#6 $\quad L : V \to V \qquad V = C[a,b]$

and L is the derivative.

$B = [1, e^x, e^{-x}]$ $\qquad$ Know: $\cosh x = \dfrac{e^x + e^{-x}}{2}$

$D = [1, \cosh x, \sinh x]$ $\qquad \sinh x = \dfrac{e^x - e^{-x}}{2}$

$$B \text{ as matrix } \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow 1\text{'s} \\ \leftarrow e^x\text{'s} \\ \leftarrow e^{-x}\text{'s} \end{array}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & -1/2 \end{bmatrix}$$

$\begin{matrix} 1 \\ 0 \\ 0 \end{matrix} = 1 \qquad \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} = e^x \qquad \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} = e^{-x}$

$\to$ $\boxed{L}$ is just the derivative

(1) A in standard is $\boxed{A = [L(e_1)\ L(e_2)\ L(e_3)]}$, $e_1 = 1$, $e_2 = e^x$, $e_3 = e^{-x}$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(2) L in our $\underbrace{[1, \cosh x, \sinh x]}_{D}$ basis as a matrix?

$$Y_D = D^{-1} A D X_D$$

$$Y_D = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & -1/2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & -1/2 \end{bmatrix}}_{\text{do this!}} X_D$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$