Math 511

Q's

3.4 #14b $\dim(\mathrm{span}(\{x, x-1, x^2+1, x^2-1\}))$

$\begin{matrix} 0 & -1 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{matrix}$

$P_3 \to$ standard basis $p_1 = 1$ $p_2 = x$ $p_3 = x^2$

$\begin{bmatrix}1\\0\\0\end{bmatrix}$ $\begin{bmatrix}0\\1\\0\end{bmatrix}$ $\begin{bmatrix}0\\0\\1\end{bmatrix}$

$p(x) = a_1 + a_2 x + a_3 x^2$

we can represent p by $\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}$

$\to$ we can turn this into what is dim for the span of $\left\{ \begin{matrix}0\\1\\0\end{matrix}, \begin{matrix}-1\\1\\0\end{matrix}, \begin{matrix}1\\0\\1\end{matrix}, \begin{matrix}-1\\0\\1\end{matrix} \right\}$

$$\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 0 & -1 & -1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & \boxed{1} & 2 \end{bmatrix}$$

$1 + x^2,\ x,\ -1 + x$

Wronskian:

$$\begin{vmatrix} x & x-1 & x^2+1 & x^2-1 \\ 1 & 1 & 2x & 2x \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0 \quad \text{(fail)}$$

$$\begin{vmatrix} x & x-1 & x^2+1 \\ 1 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2\begin{vmatrix} x & x-1 \\ 1 & 1 \end{vmatrix} = 2(x-(x-1))$$

$$= 2 \neq 0$$

lin. ind. → Matrix system of eqns $\quad \mathbb{R}^n$

$$A\mathbf{x} = \mathbf{0}$$

↔ trivial vs non-trivial

↔ non-singular vs singular

↔ $\det \neq 0$ vs $\det = 0$

$\underline{A} \to \boxed{\underline{U}}$ in reduced row echelon

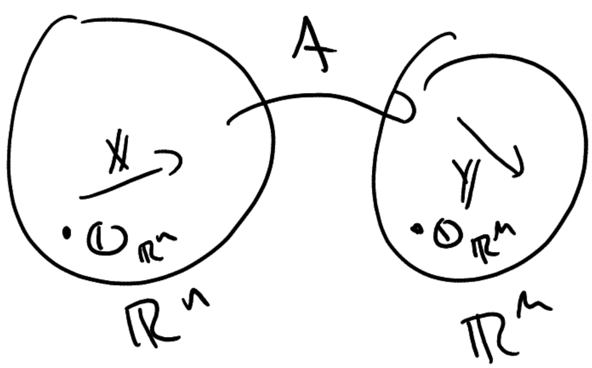
$\boxed{Q}$ Rank, Nullity,

$N(A)$, basis of $N(A)$

dep. eqns of U → dep eqns of A

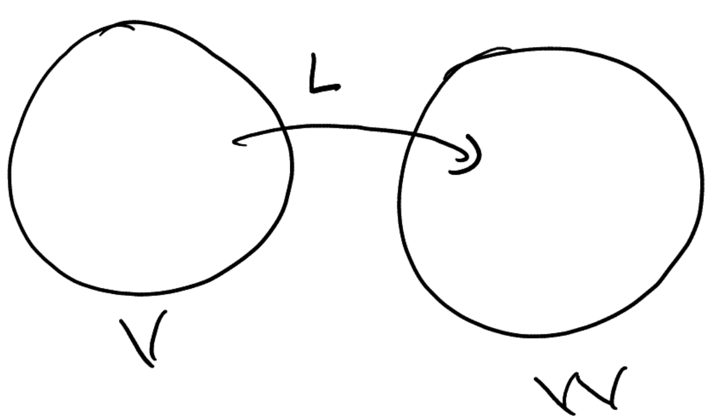
basis of row space A

basis of col. space of A



$$\underset{m \times n}{A}\mathbf{x} = \mathbf{y}$$

$\boxed{\text{Ch 4}}$



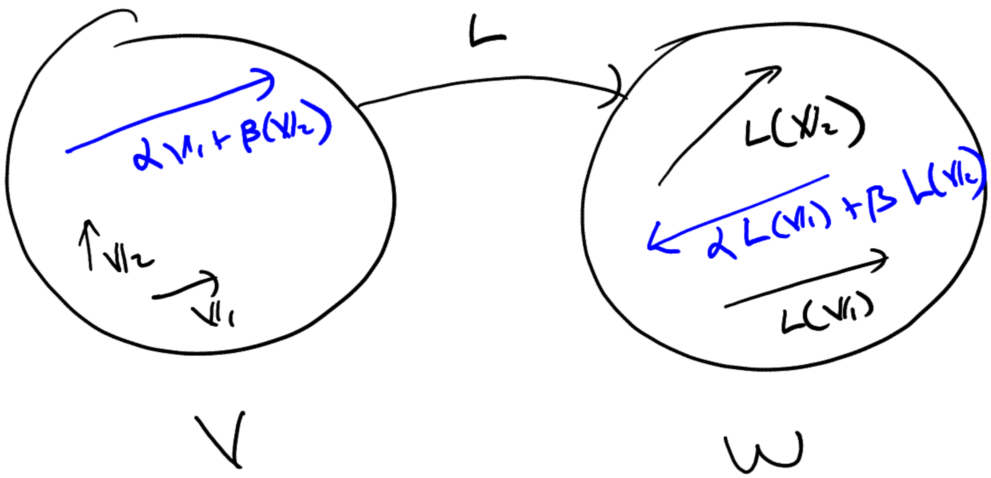
$L : V \to W$

V and W are any vector space

Narrow our study to $\boxed{\text{Linear Transforms}}$

<u>Def:</u> $L: V \to W$ is a linear transform if for all v_1, v_2 in V and α, β scalars

$$L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2)$$



Is a transformation a <u>linear transformation</u>?

Check

$\boxed{\text{tech \#1}}$ (use the def.)

$$L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2)$$

$\boxed{\text{tech \#2}}$

1. $L(\alpha v_1) = \alpha L(v_1)$
2. $L(v_1 + v_2) = L(v_1) + L(v_2)$

$\boxed{ex}$ is $L : C[a,b] \to \mathbb{R}^1$

$$L(f) = \int_a^b f(x)\,dx \quad \text{a linear transform?}$$

<u>check:</u> ① $L(\alpha V_1) = \alpha L(V_1)$

for this it means $L(\alpha f) = \alpha L(f)$

$$L(\alpha f) = \int_a^b \alpha f\,dx = \alpha \int_a^b f\,dx = \alpha L(f) \quad \checkmark$$

② $L(V_1 + V_2) = L(V_1) + L(V_2)$

for this it means $L(f+g) = \underline{L(f)} + \underline{L(g)}$

$$L(f+g) = \int_a^b (f+g)\,dx = \int_a^b f\,dx + \int_a^b g\,dx = L(f) + L(g) \quad \checkmark$$

$\boxed{ex}$ is $L : \mathbb{R}^2 \to \mathbb{R}^3$ a linear transform

where $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix}$?

$\boxed{check}$ ① $L(\alpha V_1) = \alpha L(V_1)$ $\to \alpha \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix}$

<u>for us:</u> $L\left(\alpha \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \alpha L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$

$$L\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix}\right) = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_1 + \alpha x_2 \end{bmatrix} = \alpha \begin{bmatrix} x_1 \\ x_2 \\ x_1 + x_2 \end{bmatrix} = \alpha L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \checkmark$$

② $L(V_1 + V_2) = L(V_1) + L(V_2)$

$$L\left(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}\right) = L\left(\begin{bmatrix} a+c \\ b+d \end{bmatrix}\right) = \begin{bmatrix} a+c \\ b+d \\ a+b+c+d \end{bmatrix} = \begin{bmatrix} a \\ b \\ a+b \end{bmatrix} + \begin{bmatrix} c \\ d \\ c+d \end{bmatrix}$$

show $L\left(\begin{bmatrix}a\\b\end{bmatrix}+\begin{bmatrix}c\\d\end{bmatrix}\right) = L\left(\begin{bmatrix}a\\b\end{bmatrix}\right) + L\left(\begin{bmatrix}c\\d\end{bmatrix}\right)$ ✓

Note: in general any $m \times n$ matrix A is a linear transform $L_A : \mathbb{R}^n \to \mathbb{R}^m$

where $L_A(\mathbf{x}) = A\mathbf{x}$

Show tech #1

$$L(\alpha \mathbf{v}_1 + \beta \mathbf{v}_2) = A(\alpha \mathbf{v}_1 + \beta \mathbf{v}_2)$$
$$= \alpha A \mathbf{v}_1 + \beta A \mathbf{v}_2 = \alpha L(\mathbf{v}_1) + \beta L(\mathbf{v}_2)$$ ✓

For any $L : V \to W$

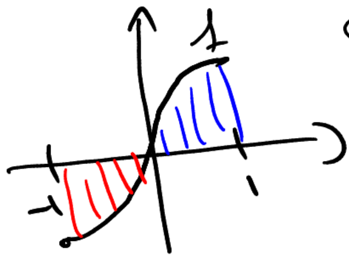
1. $L(\mathbf{0}_V) = \mathbf{0}_W$
2. $L(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n) = \alpha_1 L(\mathbf{v}_1) + \alpha_2 L(\mathbf{v}_2) + \dots + \alpha_n L(\mathbf{v}_n)$
3. $L(-\mathbf{v}) = -L(\mathbf{v})$

Def Kernel of L, denoted $\ker(L)$, is all $\mathbf{v} \in V$ that map to $\mathbf{0}_W$.

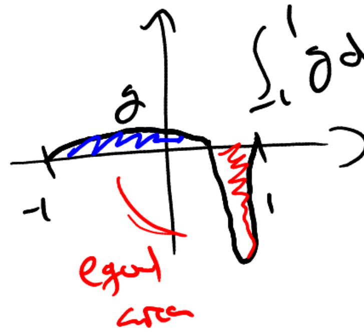
$$\ker(L) = \{ \mathbf{v} \in V \mid L(\mathbf{v}) = \mathbf{0}_W \}$$

ex $L(f) = \int_{-1}^{1} f \, dx \qquad L : C[-1,1] \to \mathbb{R}$

$\ker(L) = ?$ = all net signed area 0 functions



$\int_{-1}^{1} f\,dx = 0$



$\int_{-1}^{1} g\,dx = 0$

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② **Def** S is a subspace of X.

The image of S, denote $L(S)$, is all $w \in W$ that have a pre-image in V.

$$L(S) = \{ w \in W \mid \exists\, v \in V \text{ such that } L(v) = w \}$$

Note: $L(X)$ = range of L

Thm $\ker(L)$ is a subspace of X.
$L(S)$ is a subspace of W.

We know given $A \to L_A(x) = Ax$ is a linear transform from $\mathbb{R}^n \to \mathbb{R}^m$

Q given a linear transform from $\mathbb{R}^n \to \mathbb{R}^m$

ex $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ a+b \end{bmatrix}$ has a linear trans from $\mathbb{R}^2 \to \mathbb{R}^3$

can you find an A (a matrix) such that

$$L(x) = \underline{Ax} \text{ ?}$$

Yes!

thm given $L: \mathbb{R}^n \to \mathbb{R}^m$ then matrix A, is $m \times n$ is size, such that

$$A = [\, L(e_1) \;\; L(e_2) \cdots \;\; L(e_n) \,]$$

is the standard matrix rep. of L where $L(x) = Ax$.

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ex $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ a+b \end{bmatrix}$ $\quad L: \mathbb{R}^2 \to \mathbb{R}^3$

standard basis $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$A = \left[\, L\begin{pmatrix} 1 \\ 0 \end{pmatrix} \;\; L\begin{pmatrix} 0 \\ 1 \end{pmatrix} \,\right]$$

$$\boxed{A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}}$$

check: $A \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \\ a+b \end{bmatrix}$

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pf x in $\mathbb{R}^n$ in standard basis

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

$$\begin{aligned} L(x) &= L(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n) \\ &= x_1 L(e_1) + x_2 L(e_2) + \cdots + x_n L(e_n) \\ &= [\, L(e_1) \;\; L(e_2) \cdots \;\; L(e_n) \,] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \end{aligned}$$

$= A\mathbf{x}$

where $A = \left[\, L(\mathbf{e}_1) \;\; L(\mathbf{e}_2) \;\; \cdots \;\; L(\mathbf{e}_n) \,\right]$

$$L\left(\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}\right) \quad L\left(\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}\right)$$

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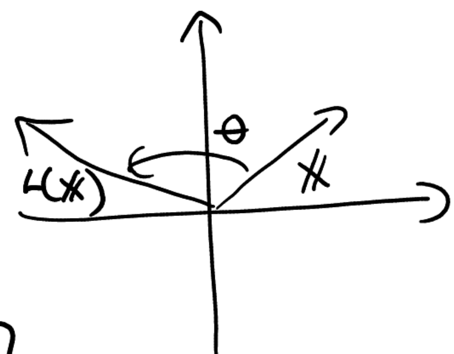
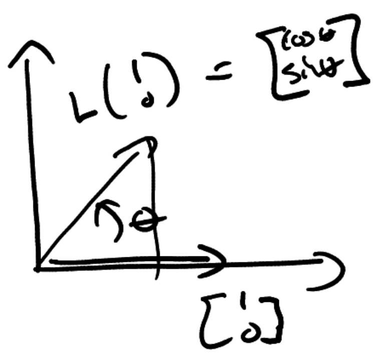
ex] <u>Note:</u> A the standard matrix rep. of $L : \mathbb{R}^n \to \mathbb{R}^n$ is simply $A = \left[\, L(\mathbf{e}_1) \;\; L(\mathbf{e}_2) \;\; \cdots \;\; L(\mathbf{e}_n) \,\right]$

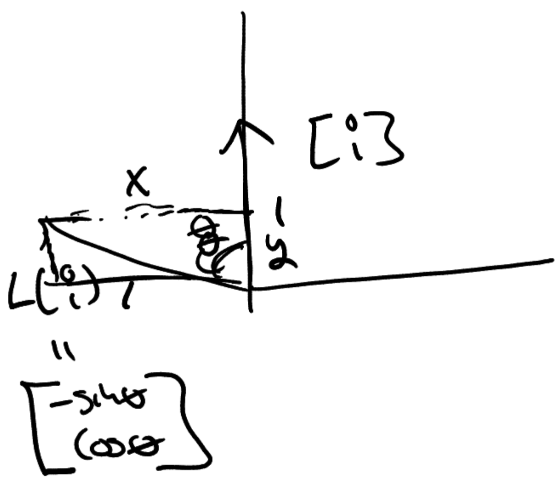
ex] rotations by θ in $\mathbb{R}^2$ are a linear transformation

$L : \mathbb{R}^2 \to \mathbb{R}^2$



$$A = \left[\, L\begin{pmatrix}1\\0\end{pmatrix} \;\; L\begin{pmatrix}0\\1\end{pmatrix} \,\right]$$





$$\rightarrow \boxed{A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$$
