Math 511

Q's given two basis sets $B = \{b_1, b_2, \dots, b_k\}$

$D = \{d_1, d_2, \dots, d_k\}$

for vector space, V, with $\dim(V) = k$

given $[c]_B$ $[c]_D$ ← coordinates

ex $v \in V$

$$v = \underbrace{c_1 b_1 + c_2 b_2 + \dots + c_k b_k}$$

coordinates are the uniq coeff. to form v from b_i

Conversion

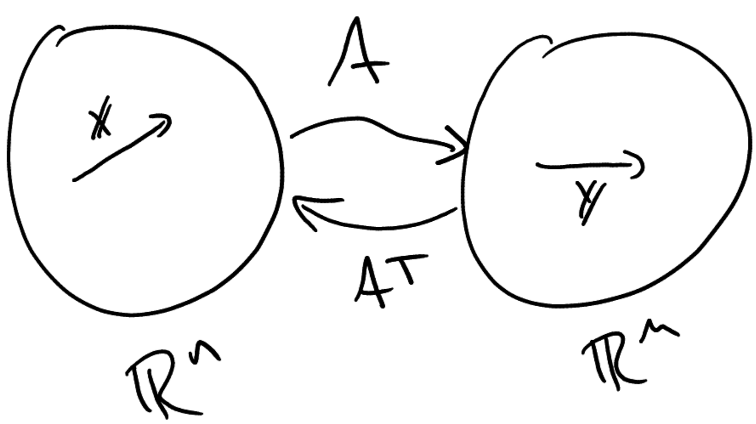
① $B[c]_B = [c]_E$ ← standard coord.

② $[c]_B = B^{-1}[c]_E$

③ B and D

$$B[c]_B = D[c]_D$$

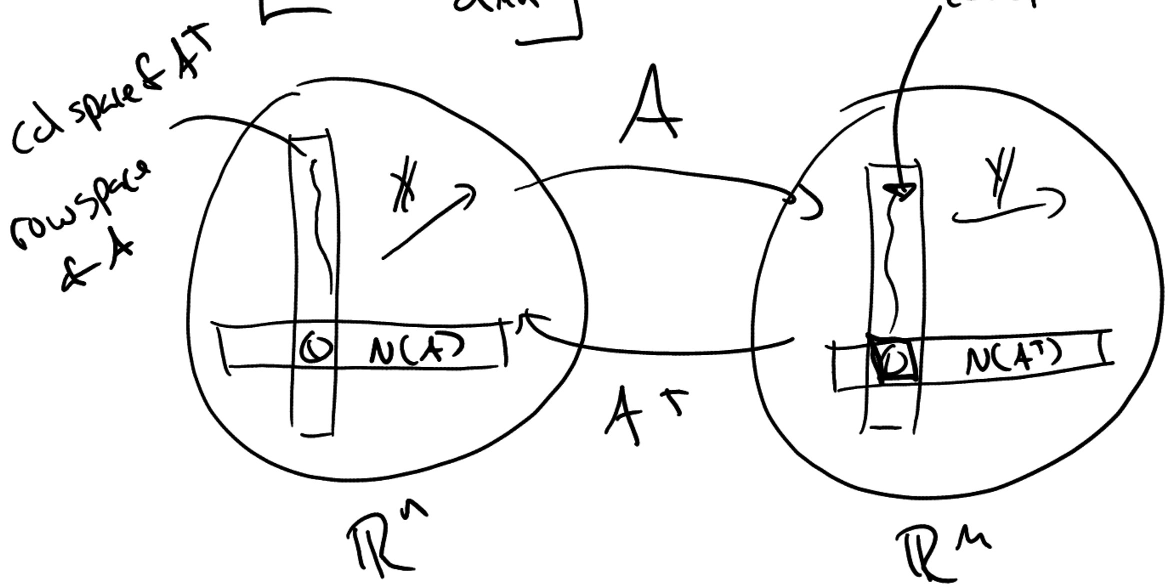
3.6

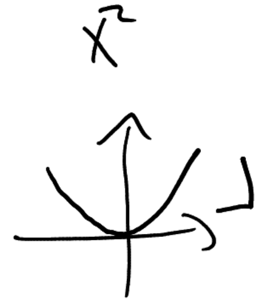


$$\underset{(m\times n)}{A}\ \underset{(n\times 1)}{x} = \underset{(m\times 1)}{y}$$

$$\underset{(n\times m)}{A^T}\ \underset{(m\times 1)}{y} = \underset{(n\times 1)}{x}$$

$$\underset{m\times n}{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn}\end{bmatrix}$$





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in $\boxed{3.5}$ change of basis

matrix $B = [b_1\ b_2 \cdots b_k]$

$\boxed{\text{ex}}$ $\mathbb{R}^3$ $\quad B = \begin{bmatrix}1&0&1\\0&1&0\\0&1&1\end{bmatrix}$

$$\boxed{v} = \boxed{3}\begin{bmatrix}1\\0\\0\end{bmatrix} + \boxed{1}\begin{bmatrix}0\\1\\1\end{bmatrix} + \boxed{(-2)}\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$[v]_B = \begin{bmatrix}3\\1\\-2\end{bmatrix}_B$$

$$\begin{bmatrix}1&0&1\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}3\\1\\-2\end{bmatrix}_B = \begin{bmatrix}1\\1\\-1\end{bmatrix}$$

Given **any** matrix $\underset{m \times n}{A} = [a_1, a_2 \dots a_n]$

$\text{Span}(a_1, a_2 \dots a_n) = c_1 a_1 + c_2 a_2 + \dots + c_n a_n = v$

subspaces of $\mathbb{R}^{m \times 1} = \mathbb{R}^m$

$\boxed{\text{Def}}$ col space of A is span(A's col. vectors)

Also $A = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{bmatrix}$

$\text{Span}(\vec{a}_1, \vec{a}_2 \dots \vec{a}_m) = c_1 \vec{a}_1 + c_2 \vec{a}_2 + \dots + c_m \vec{a}_m = \underset{1 \times n}{\vec{v}}$

subspace of $\mathbb{R}^{1 \times n}$

$\boxed{\text{ex}}$ $A = \begin{array}{c} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix} \\ x_1 \ \ x_2 \ \ x_3 \end{array} \rightarrow \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U$

1. lead vars : x_1, x_2 $\qquad |\text{lead}| + |\text{free}| = |\#\text{ of cols}|$
2. free vars : $\boxed{x_3}$
3. $N(A) \rightarrow$ solve $Ax = 0$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 2 & 1 & 4 & 0 \\ 4 & 7 & 8 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = a$

$x_2 = 0$

$x_1 = -2a$

$N(A)$ all vectors $a \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

Study Col. Spaces and Row Spaces

$\boxed{\text{Th}^m\ 3.6.1}$ row equiv matrices have the same row space.

$\boxed{\text{ex}}$ $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix} \xrightarrow[\text{ops}]{\text{row}} \boxed{\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U}$

row space of $A = a_1[1\ 3\ 2] + a_2[2\ 1\ 4] + a_3[4\ 7\ 8]$

row space of $U = a_1[1\ 3\ 2] + a_2[0\ 1\ 0] + \boxed{\cancel{a_3[0\ 0\ 0]}}$

$\boxed{\text{So}}$ dim(row space of U) = dim(row space of A)

$\boxed{\text{and}}$ dim(row space of U) = # of lead vars

$\boxed{\text{Def}}$ dim (row space of A) = # of lead vars = r

call this $\boxed{\text{rank}(A) = r}$

Stuff we now know $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U$

rank$(A) = 2$ $\quad |\text{lead}| = 2 \quad |\text{free}| = 1$

$\dim(N(A)) = 1$

Col Space

$$\underset{m\times n}{A} = [\mathbb{a}_1\ \mathbb{a}_2\ \cdots\ \mathbb{a}_n]$$

col space of A = $\underline{c_1 \mathbb{a}_1 + c_2 \mathbb{a}_2 + \cdots + c_n \mathbb{a}_n}$

Note: $A\mathbb{x} = \mathbb{b}$

$$x_1 \mathbb{a}_1 + x_2 \mathbb{a}_2 + \cdots + x_n \mathbb{a}_n = \mathbb{b}$$

Thm $A\mathbb{x} = \mathbb{b}$ has a soln. iff $\mathbb{b} \in$ col space of A.

Thm A is $m\times n$

① $A\mathbb{x} = \mathbb{b}$ has a solution for all $\mathbb{b} \in \mathbb{R}^m$ iff the col. vectors span $\mathbb{R}^m$

② Also, the system $A\mathbb{x} = \mathbb{b}$ has at most one soln for every $\mathbb{b} \in \mathbb{R}^m$ iff the vectors are linearly ind.

Corollary A $n\times n$ is non-singular iff col. vectors of A form a basis for $\mathbb{R}^n$

Rank-Nullity

Def: rank(A) = dim of row space = # of lead

Def: nullity(A) = dim of $N(A)$ = # of free vars

rank + nullity = n (# of columns)

Dependency (linearly ind. vs linearly dep vectors)

$v_1 = \begin{matrix}1\\2\\4\end{matrix} \quad v_2 = \begin{matrix}3\\1\\7\end{matrix} \quad v_3 = \begin{matrix}2\\4\\8\end{matrix}$

does $c_1 \begin{bmatrix}1\\2\\4\end{bmatrix} + c_2 \begin{bmatrix}3\\1\\7\end{bmatrix} + c_3 \begin{bmatrix}2\\4\\8\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

have <u>only</u> trivial $c = \begin{bmatrix}0\\0\\0\end{bmatrix}$ or non-trivial?

$\rightarrow \begin{vmatrix}1 & 3 & 2\\2 & 1 & 4\\4 & 7 & 8\end{vmatrix} \begin{matrix} = 0? \\ \neq 0? \end{matrix}$

$\rightarrow$ just use $|U| = \begin{vmatrix}1 & 0 & 2\\0 & 1 & 0\\0 & 0 & 0\end{vmatrix} = 0 \rightarrow$ A is singular

$\boxed{\text{dependency equations of } U}$

$U = [u_1 \; u_2 \; u_3] = \begin{bmatrix}1 & 0 & 2\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$

free cols.

$\begin{bmatrix}1 & 3 & 2\\2 & 1 & 4\\4 & 7 & 8\end{bmatrix} \rightarrow \begin{bmatrix}1 & 3 & 2\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix} \rightarrow \begin{bmatrix}1 & 0 & 2\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$

(gauss) (gauss-jordan)

$\rightarrow$ dependent on lead cols.

$u_3 = 2u_1$

$\boxed{\text{Note:}}$ dep. eqn's of U (reduced row-ech) are dep eqn's of A as well

$\rightarrow \quad a_3 = 2a_1$

$$A_{3\times 5} = \begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 2 & 4 & 5 & 5 & 4 \\ 3 & 6 & 7 & 8 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{c} \begin{bmatrix} 1 & \boxed{2} & 0 & \boxed{5} & \boxed{-3} \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \quad\quad a_2 \quad\quad a_4 \;\; a_5 \end{array} = U$$

① $\text{rank}(A) = 2$

② $\text{nullity}(A) = 3$

③ $N(A)$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 5 & -3 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_2 = a$ $\quad x_3 = b - 2c$

$x_4 = b$ $\quad x_1 = -2a - 5b + 3c$

$x_5 = c$

$$x = \begin{bmatrix} -2a - 5b + 3c \\ a \\ b - 2c \\ b \\ c \end{bmatrix} = a\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 3 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

④ Dep. eqn's

$a_2 = 2a_1$

$a_4 = 5a_1 - a_3$

$a_5 = -3a_1 + 2a_3$

$$\boxed{\begin{aligned} a_2 &= 2a_1 \\ a_4 &= 5a_1 - a_3 \\ a_5 &= -3a_1 + 2a_3 \end{aligned}}$$

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 2 & 4 & 5 & 5 & 4 \\ 3 & 6 & 7 & 8 & 5 \end{bmatrix}$$

⑤ basis for row space $= [1 \; 2 \; 0 \; 5 \; -3]$
$[0 \; 0 \; 1 \; -1 \; 2]$

basis for col space $= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$