Math 511

Q's/ 3.2 5. $\boxed{P_4} \to p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

a_i are any real number

Def: $(p+q)(x) = p(x) + q(x)$

$(\alpha p)(x) = \alpha p(x)$

$\to 0 : \; z(x) = 0 + 0x + 0x^2 + 0x^3$

5a) $S = \{ \boxed{\text{the even polynomials}} \}$ → Sym. about y-axis → $e(-x) = e(x)$

① $z(x) \in S$ true

② if p, q are even $(p+q) = ?$

check $(p+q)(-x) = p(-x) + q(-x) = p(x) + q(x)$

$= (p+q)(x)$

so $p+q$ is even

③ check if αp is even.

(you do this)

5b) $S = \{ \text{all polys of degree } 3 \}$

ex's $p(x) = x^3$

$p(x) = 1 - \pi x^3$

① is $z(x) = 0 + 0x + 0x^2 + 0x^3$ a deg 3 poly?

$\boxed{\text{No}}$ → so S is not a subspace.

$\text{Span}(\{\mathbb{V}_1, \mathbb{V}_2, .., \mathbb{V}_k\}) = V$ ← a vector space

any $\mathbb{V}$ in V can be written as

$$\mathbb{V} = \alpha_1 \mathbb{V}_1 + \alpha_2 \mathbb{V}_2 + \dots + \alpha_k \mathbb{V}_k$$

Q's

1. uniq? → linear ind. vs dep.
2. best? → 'standard' vectors
3. fewest? → minimal spanning set

Linearly independent.

dependent eqn

ex $\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3$ but notice $\boxed{\mathbb{V}_3 = \mathbb{V}_1 - 2\mathbb{V}_2}$

$\text{Span}(\mathbb{V}_1, \mathbb{V}_2, \mathbb{V}_3)$ any $\mathbb{V} = \underline{\alpha_1} \mathbb{V}_1 + \underline{\alpha_2} \mathbb{V}_2 + \underline{\alpha_3} \mathbb{V}_3$

but $\alpha_1 \mathbb{V}_1 + \alpha_2 \mathbb{V}_2 + \alpha_3 (\mathbb{V}_1 - 2\mathbb{V}_2)$

$$= \underbrace{(\alpha_1 + \alpha_3)}_{\text{const.}} \mathbb{V}_1 + \underbrace{(\alpha_2 - 2\alpha_3)}_{\text{const.}} \mathbb{V}_2$$ ← $\text{Span}(\mathbb{V}_1, \mathbb{V}_2)$

Q when does a dep. eqn exist?

ex $\mathbb{V}_1 - 2\mathbb{V}_2 - \mathbb{V}_3 = 0$

$(1)\mathbb{V}_1 + (-2)\mathbb{V}_2 + (-1)\mathbb{V}_3 = 0$

$$\boxed{\begin{bmatrix} \mathbb{V}_1 & \mathbb{V}_2 & \mathbb{V}_3 \end{bmatrix}} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} = 0$$

Def $\mathbb{V}_1, \mathbb{V}_2, \dots, \mathbb{V}_k$ are linearly independent

iff

$$c_1 \mathbb{V}_1 + c_2 \mathbb{V}_2 + \dots + c_k \mathbb{V}_k = \mathbb{O}$$

implies $c_i = 0$ for all i. ← trivial soln.

Def $\mathbb{V}_1, \mathbb{V}_2, \dots, \mathbb{V}_k$ are linearly dep.

iff $c_1 \mathbb{V}_1 + \dots + c_k \mathbb{V}_k = \mathbb{O}$

implies there is a non-trivial soln for c_i

Note. let $A = [\mathbb{V}_1 \ \mathbb{V}_2 \dots \mathbb{V}_k]$

$$\boxed{c_1 \mathbb{V}_1 + c_2 \mathbb{V}_2 + \dots + c_k \mathbb{V}_k = \mathbb{O}}$$

→ trivial soln only? $\boxed{\text{ind}}$

→ non-trivial soln? $\boxed{\text{dep}}$

is just $A \mathbb{C} = \mathbb{O}$

$$\mathbb{C} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

in $\mathbb{R}^n$ vector space we can make the

$$[\mathbb{V}_1 \ \mathbb{V}_2 \dots \mathbb{V}_k] \mathbb{C} = \mathbb{O} \qquad \text{system of eqn's}$$

call $A = [\mathbb{V}_1 \ \mathbb{V}_2 \dots \mathbb{V}_k]$

Ind → has only trivial soln we call A non-singular → $\det(A) \neq 0$

dep → has a non-trivial soln we call A singular → $\det(A) = 0$

So: $\mathbb{V}_1, \mathbb{V}_2, \ldots, \mathbb{V}_k$ in $\mathbb{R}^n$ are

linearly ind if $\det([\mathbb{V}_1\ \mathbb{V}_2 \cdots \mathbb{V}_k]) \neq 0$

linearly dep if $\det([\mathbb{V}_1\ \mathbb{V}_2 \cdots \mathbb{V}_k]) = 0$

$\boxed{\text{th}^m\ 3.3.2}$

for $\mathbb{V} \in \text{Span}(\{\mathbb{V}_1, \ldots \mathbb{V}_k\})$

$$\mathbb{V} = c_1 \mathbb{V}_1 + c_2 \mathbb{V}_2 + \cdots + c_k \mathbb{V}_k$$

the values for $c_1, c_2, \ldots, c_k$ are uniq.

iff $\mathbb{V}_i$ are linearly ind.

$\mathbb{R}^n$ → use det to find if linearly ind/dep.

$\underbrace{P_n?.\ C[a,b]?.}$

↳ go back to solving

$$\boxed{c_1 \mathbb{V}_1 + c_2 \mathbb{V}_2 + \cdots + c_k \mathbb{V}_k = \mathbb{0}}$$

for c_i.

ex: are $1+x$, x^2-2, x linearly ind?

in P_3

$$\begin{cases} c_1(1+x) + c_2(x^2-2) + c_3(x) = 0 + 0x + 0x^2 \\ (c_1 - 2c_2) + (c_1 + c_3)x + c_2 x^2 = 0 + 0x + 0x^2 \end{cases}$$

$$\left.\begin{array}{l} c_1 - 2c_2 = 0 \\ c_1 + c_3 = 0 \\ c_2 = 0 \end{array}\right\} \to c_1 = 0,\ c_2 = 0,\ c_3 = 0$$

only the trivial soln.

<u>so</u> $\boxed{1+x,\ x^2-2,\ x \text{ are linearly ind.}}$

C[-1,1] $\quad 1, x, e^x$ are these linearly ind?

$$\boxed{c_1 v_1 + c_2 v_2 + c_3 v_3 = 0} \checkmark$$

$$c_1(1) + c_2(x) + c_3(e^x) = 0$$

<u>Modify the problem</u>

original $\quad c_1(1) + c_2(x) + c_3(e^x) = 0$

1st Deriv. $\quad c_1(1)' + c_2(x)' + c_3(e^x)' = (0)'$

2nd Deriv $\quad c_1(1)'' + c_2(x)'' + c_3(e^x)'' = (0)''$

$$\begin{bmatrix} 1 & x & e^x \\ (1)' & (x)' & (e^x)' \\ (1)'' & (x)'' & (e^x)'' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x \in [-1,1]$

$$\begin{bmatrix} 1 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{bmatrix} C = 0$$

if we have at least one x such that at that place the matrix is $\boxed{\text{non-singular}}$ then the functions are ind. at that x.

$$\det\begin{bmatrix} 1 & x & e^x \\ 0 & 1 & e^x \\ 0 & 0 & e^x \end{bmatrix} = \boxed{e^x}$$

is this ever non-zero?

yes → linearly ind.

no → ?, ?, ? test fails.

Def Wronskian

$$\det\left(\begin{bmatrix} f_1 & f_2 & \cdots & f_k \\ f_1' & f_2' & \cdots & f_k' \\ \vdots & \vdots & & \vdots \\ f_1^{(k-1)} & \cdots & \cdots & f_k^{(k-1)} \end{bmatrix}\right) = w(x)$$

if $w(x)$ is ever non-zero → linearly ind.

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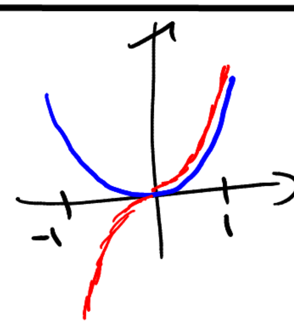
ex x^2, $x|x|$ in $C[-1,1]$



$$w(x) = \begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix}$$

$$(x|x|)' = \left(\begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}\right)'$$

$$(x|x|)' = \begin{cases} 2x & x \geq 0 \\ -2x & x < 0 \end{cases}$$

$$= 2x^2|x| - 2x^2|x|$$

$$= 2\begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$= 0$$

$$= 2|x|$$

Wronskian doesn't help us know anything.

try $\boxed{c_1 x^2 + c_2 x|x| = 0}$

pick some x's between -1 and 1 and check for ind.

pick $x=-1$ and $x=1$

@ $x=-1$
@ $x=1$

$$\begin{cases} c_1 - c_2 = 0 \\ c_1 + c_2 = 0 \end{cases}$$

$$2c_1 = 0$$

$c_1 = 0 \to c_2 = 0$ only trivial soln.

$\boxed{\text{ind.}}$

$\boxed{3.4}$ Basis / Dimension

Def: we call $v_1, v_2, \ldots, v_k$ a basis of V

iff
① $v_1, v_2, \ldots, v_k$ are linearly ind.
② $\text{span}(v_1, v_2, \ldots, v_k) = V$

Def ① if V has a basis of n-vectors
we call n = dimension of V

$$\dim(V) = n$$

② $\dim(\{0\}) = 0$

③ $\dim(V) = n \in \mathbb{N}$
we call V finite dimensional