Math 321

~~Q's~~ 1.4 #12

just multiply both ways and get I

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \frac{a_{22}}{a_{11}a_{22}-a_{12}a_{21}} & \text{etc} \end{pmatrix} =$$

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example #4 p. 63

new $r_2 = r_2 + (1) r_1$ $\to$ $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

new $r_3 = r_3 + (-2) r_1$ $\to$ $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right]$$

$Nr_3 = r_3 + (3) r_2$

$$\to \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right]$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

$$A = L \begin{pmatrix} 1 & 4 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 6 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 6 \end{pmatrix}$$

$Nr_2 = r_2 + (-\frac{1}{2}) r_3$

$Nr_1 = r_1 + (-\frac{1}{2}) r_3$

$$\to \left[\begin{array}{ccc|ccc} 1 & 4 & 0 & \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 2 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right]$$

? $E_5 E_4 E_3 E_2 E_1 A = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$?

Determinates

Def: determinates by cofactor expansion

(expand along row 1) A is $n \times n$

$$\det(A) = \begin{cases} a_{11} & \text{if } n=1 \\ a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} & \text{if } n>1 \end{cases}$$

A_{ij} are a_{ij}'s cofactors

ex $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 2 & 1 & 4 \end{bmatrix}$

sign of cofactors $\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$

$$\det(A) = 1\,A_{11} + 2\,A_{12} + 3\,A_{13}$$

$$= 1\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} + 2(-1)\begin{vmatrix} 0 & 1 \\ 2 & 4 \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} = ? = 1$$

Note: we can expand along <u>any</u> row or col.

(row 2 is "smarter" for our example)

$$\det(A) = 0\cdot(-1)\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} + 1\cdot(+1)\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} + 1(-1)\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= 0 + (-2) + (3) = 1$$

Note on A_{ij} (cofactors)

→ you could collect all the cofactors for each a_{ij} for "fun"

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \longrightarrow \begin{bmatrix} A_{11} & A_{12} & & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \\ A_{n1} & \cdots & \cdots & A_{nn} \end{bmatrix} = adj(A)$$

call this the adjoint of A

Cost? (CPU time?)

$$\det(A) = \underline{a_{11}} \cdot \underline{A_{11}} + \underline{a_{12}} \cdot \underline{A_{12}} + \cdots + \underline{a_{1n}} \cdot \underline{A_{1n}}$$

$\underline{n\text{-mult.}}$

$A_{11} = (-1)^{1+1} |M_{11}|$

$(n-1)$ mult

$\det(A)$ on order of $n!$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad 2! = 2$$

$A_{3\times 3}$ $\quad 3! = 6$

$n=4 \quad 4! = 24$

$n=5 \quad 5! = 120$

$n=6 \quad 6! = 720$

$n=7 \quad 7! = 5040$

Faster way?

→ remember 0's are the smarter cofactor expansion

Triangular:

$$T = \begin{bmatrix} t_{11} & t_{12} & \cdot & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ 0 & 0 & t_{33} \cdots & t_{3n} \\ & & \ddots & \\ 0 & 0 & 0 \cdots 0 & t_{nn} \end{bmatrix} \qquad \det(T) = t_{11} \begin{vmatrix} t_{22} & \cdots & t_{2n} \\ 0 & \ddots & t_{nn} \end{vmatrix} + 0$$

continue

$$\boxed{\det(T) = t_{11} t_{22} t_{33} \cdots t_{nn}}.$$

Ex: A is 30×30

$$A = \begin{bmatrix} 1 & 2 & 3 & \cdots & 30 \\ 0 & 2 & 3 & \cdots & 30 \\ 0 & 0 & 3 & \cdots & 30 \\ \vdots & & & & \\ 0 & 0 & \cdots & 0 & 30 \end{bmatrix} = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot 30 = \boxed{30!}$$

So

given $A \xrightarrow[\text{Elim. matrices}]{\text{row ops}} U$ (upper triangular)

1. [$\det(A)$ $\mathcal{O}(n!)$ cost] expensive
2. [$A \to U$ (gauss elim) is $\mathcal{O}(n^3)$
   $\det(U) = u_{11} u_{22} \cdots u_{nn}$] cheap

Figure out $\det(E_k E_{k-1} \cdots E_2 E_1 A) = \det(U)$

$\det(A) = ?$

$\boxed{\text{So}}$ (1) $\det(E_{type1} A) = ?$

(2) $\det(E_{type2} A) = ?$

(3) $\det(E_{type3} A) = ?$

$\boxed{\text{Type1}}$ (row swap)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(A) = ad - bc$$

$$\det(E_{type1} A) = \det\left(\begin{bmatrix} c & d \\ a & b \end{bmatrix}\right) = bc - ad$$

row swap A

$E_{type1} A$

<u>Notice</u> $\det(A)$ vs $\det(E_{type1} A)$

$$\det(E_{type1} A) = -\det(A)$$

$\boxed{\text{So}}$ $\det(E_{type1} A) = (-1)\det(A)$

<u>use</u> this to find

$$\det(E_{type1}) = \det(E_{type1} \cdot I)$$
$$= (-1)\det(I) = -1$$

and $\det(E_{type1} A) = \det(E_{type1})\det(A)$

type2 $\det(E_{type2}\, A) = ?$

1st $\det(E_{type2}) = m$

$$\det(E_{type2}\, A) = (m\, a_{k1}) A_{k1} + (m\, a_{k2}) A_{k2} + \dots + (m\, a_{kn}) A_{kn}$$

cofactor along row that's multiplied by m (row k)

so $\det(E_{type2}\, A) = m \det(A)$

and $\det(E_{type2}\, A) = \det(E_{type2}) \det(A)$

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type3 $\det(E_{type3}\, A) = ?$

1st $\det(E_{type3}) = 1$

$$\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ m & & 1 \end{bmatrix}$$

$$\det(E_{type3}\, A) = (a_{i1} + m\, a_{j1}) A_{i1} + (a_{i2} + m\, a_{j2}) A_{i2} + \dots + (a_{in} + m\, a_{jn}) A_{in}$$

put an m in a_{ij} position of I

$$= \det(A) + m\,(a_{j1} A_{i1} + \dots + a_{jn} A_{in})$$

New row$_i$ = row$_i$ + m row$_j$ ← use this row for cofactor expansion

$\boxed{\text{lemma}}$ $a_{j1}A_{i1} + a_{j2}A_{i2} + \cdots + a_{jn}A_{in}$

$= \det(A)$ if $i=j$ (same row of A as cofactor row)

$= 0$ if $i \neq j$ (different rows)

So the above is

$$\det(E_{\text{type}3}\, A) = \det(A) + 0$$

$\boxed{\text{or}}$

$$\det(E_{\text{type}3}\, A) = (1)\det(A)$$

$$\det(E_{\text{type}3}\, A) = \det(E_{\text{type}3})\det(A)$$

$\boxed{\text{Summary}}$

$$\det(E \cdot A) = \det(E)\det(A)$$

and

$$\det(E_{\text{type}\,i}) = \begin{cases} 1 & \text{type 3} \\ -1 & \text{type 1} \\ m & \text{type 2} \end{cases}$$

$\boxed{\text{ex}}$ A is 5×5

$$A \longrightarrow U = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & 0 & 1 & 2 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\left(E_7 \, E_6 \, E_5 \, E_4 \, E_3 \, E_2 \, E_1 \, A\right) = U$$

mult by 3 (E_2)

E_7: mult. by $1/10$; E_6: mult by $1/2$; E_5: row swap; E_4: type 3; E_3: type 3; E_1: row swap

$$\left(\tfrac{1}{10}\right)\left(\tfrac{1}{2}\right)(-1)(1)(1)(3)(-1)\det(A) = -6$$

$$\det(A) = \frac{-6}{3} \cdot 20 = \underline{\underline{-40}}$$
