# Math 511

Q's/ 1.4 #19   if $A^2 = 0$ ← all zero matrix

show Inx   then   show $(I - A)$ is non-singular ← show $(I-A)$ has an inverse

and $(I-A)^{-1} = I + A$

(really says to show $(I-A)(I+A) = I$

and $(I+A)(I-A) = I$ )

$$\begin{aligned}(I-A)(I+A) &= (I-A)I + (I-A)A \\ &= (I-A) + IA - A\cdot A \\ &= I - A + A - A^2 \\ &= I - A + A + 0 \\ &= I\end{aligned}$$

---

1.5  Find $A^{-1}$ and $A = LU$

Note: lower triangular matrix $\begin{bmatrix} \ddots & 0 \\ \#\text{'s} & \ddots \end{bmatrix}$

ex $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -3 & 4 \end{bmatrix}$

upper triangular matrix $\begin{bmatrix} \ddots & \#\text{'s} \\ 0 & \ddots \end{bmatrix}$

ex $\begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

diagonal matrix (both lower and upper)

$$\begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix} \quad \text{ex} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

---

$\boxed{\text{ex}}$

$$A = \begin{bmatrix} -1 & -3 & -3 \\ 2 & 7 & 1 \\ 3 & 8 & 3 \end{bmatrix}$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} -1 & -3 & -3 & 1 & 0 & 0 \\ 2 & 7 & 1 & 0 & 1 & 0 \\ 3 & 8 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} -1 & -3 & -3 & 1 & 0 & 0 \\ 0 & 1 & -5 & 2 & 1 & 0 \\ 0 & -1 & -6 & 3 & 0 & 1 \end{array}\right]$$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & \end{bmatrix}$$

$r_3 + r_2 = \text{New } r_3$

$$\boxed{\left[\begin{array}{ccc|ccc} -1 & -3 & -3 & 1 & 0 & 0 \\ 0 & 1 & -5 & 2 & 1 & 0 \\ 0 & 0 & -11 & 5 & 1 & 1 \end{array}\right]}$$

but $E_1^{-1} E_2^{-1} E_3^{-1}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix}$$

$\boxed{\text{so}}$

$$E_3 E_2 E_1 A = \begin{bmatrix} -1 & -3 & -3 \\ 0 & 1 & -5 \\ 0 & 0 & -11 \end{bmatrix}$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} \begin{bmatrix} -1 & -3 & -3 \\ 0 & 1 & -5 \\ 0 & 0 & -11 \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ \boxed{-2} & 1 & 0 \\ \boxed{-3} & \boxed{-1} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} -1 & -3 & -3 \\ 0 & 1 & -5 \\ 0 & 0 & -11 \end{bmatrix}}_{U}$$

upto 1.4 show $A^{-1}$ by $AA^{-1} = A^{-1}A = I$

1.5 find $A^{-1}$ by $[A \mid I] \xrightarrow{\text{apply elem matrices}} [I \mid A^{-1}]$

ch 2 When does $A$ have an inverse?

---

before That extend $A\mathbf{x} = \begin{bmatrix} \vec{a}_1 \mathbf{x} \\ \vec{a}_2 \mathbf{x} \\ \vdots \\ \vec{a}_m \mathbf{x} \end{bmatrix}$

or $A = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{bmatrix}$ then $A\mathbf{x} = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{bmatrix} \mathbf{x} = \begin{bmatrix} \vec{a}_1 \mathbf{x} \\ \vec{a}_2 \mathbf{x} \\ \vdots \\ \vec{a}_m \mathbf{x} \end{bmatrix}$

and $\underset{m \times n}{A}\ \underset{n \times k}{B} = \underset{m \times 1}{\begin{bmatrix} \vec{a}_1 \\ \hline \vec{a}_2 \\ \hline \vdots \\ \hline \vec{a}_m \end{bmatrix}} \underset{1 \times k}{\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{bmatrix}}$

→ what about other partitions?

ex

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 4 & 1 \\ 1 & 1 \end{bmatrix}$$

$3 \times 3$ $\qquad$ $3 \times 2$

$$\underset{2\times 2}{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} \end{bmatrix}}$$

---

Scalar product $\quad$ (row vector)(col vector) = Scalar

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 4 + 10 + 18 = 32$$

outer product

$$\underset{3\times 1}{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}} \underset{1\times 3}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}} = \begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{bmatrix}$$

---

Matrix Outer product Expansion

(col) (row)

given $X_{m \times n}$ $\quad$ $Y_{k \times n}$

$X = [\mathbb{x}_1 \ \mathbb{x}_2 \cdots \mathbb{x}_n] \quad Y = [\mathbb{y}_1 \ \mathbb{y}_2 \cdots \mathbb{y}_n]$

Consider: $X Y^T = [\mathbb{x}_1 \ \mathbb{x}_2 \cdots \mathbb{x}_n] \begin{bmatrix} \mathbb{y}_1^T \\ \mathbb{y}_2^T \\ \vdots \\ \mathbb{y}_n^T \end{bmatrix}$

$1 \times n$ partition

$n \times 1$ partition

$X Y^T = \mathbb{x}_1 \mathbb{y}_1^T + \mathbb{x}_2 \mathbb{y}_2^T + \cdots + \mathbb{x}_n \mathbb{y}_n^T$

---

Ch 2

when does $A^{-1}$ exist?

Find a test (metric) that will allow us to say

$A$ has an inverse?

vs $A$ doesn't have an inverse?

---

Play around finding $A^{-1}$ and make a guess.

① $A$ is $1 \times 1$ $\quad A = [a]$

Find $A^{-1}$ $\quad [a \mid 1]$

$[1 \mid 1/a]$ so $A^{-1} = [1/a]$

Notice that $A^{-1}$ exists as long as $a \neq 0$

② $A$ is $2\times 2$ $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

find $A^{-1}$ $\quad \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$

$$\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ c & d & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & d-\frac{bc}{a} & -c/a & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & \frac{ad-bc}{a} & -c/a & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

Notice $A^{-1}$ exists as long as $\boxed{ad-bc \neq 0}$

③ $A$ is $3\times 3$ $\quad \left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$

<u>do the work..</u>

$$A^{-1} \text{ exists if } \overbrace{a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23}}^{a_{11}(a_{22}a_{33} - a_{32}a_{23})}$$
$$-[a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23}]$$
$$+[a_{13}\underline{a_{21}a_{32}} - a_{13}\underline{a_{31}a_{22}}] \neq 0$$

---

to say $A^{-1}$ exists we are going to find the determinant of $A$

<u>Notation:</u> $\det(A) = |A|$

if $\det(A) = 0$ $A$ is singular and $A^{-1}$ doesn't exist

if $\det(A) \neq 0$ $A$ is non-singular $A^{-1}$ exists

$\det(A)$ is inductively defined.

---

$A$ is $n \times n$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \\ & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Terms: Matrix $M_{ij}$ is all the values of $A$ after you remove the $i^{th}$ row and $j^{th}$ col.

ex
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad a_{23} \quad \text{so} \quad M_{23} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$$

① Call $\det(M_{ij})$ the minor of $a_{ij}$

② Call cofactor of $a_{ij}$ to be $= (-1)^{i+j} |M_{ij}|$

Notation: $A_{ij} = (-1)^{i+j} |M_{ij}|$

cofactor expansion along 1st row

$$\det(A) = a_{11} \underset{(n-1)\times(n-1)}{\underline{\underline{A_{11}}}} + a_{12} \underline{\underline{A_{12}}} + \dots + a_{1n} \underline{\underline{A_{1n}}}$$