Math 322

9.1 R is a relation or binary relation is subset of $A \times B$

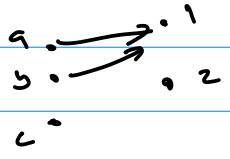
if $(a,b) \in R$ use aRb

$(a,b) \notin R$ use $a \not R b$

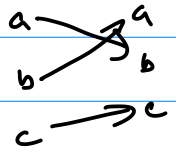
Def: R is a relation on set A

→ subset of $A \times A$

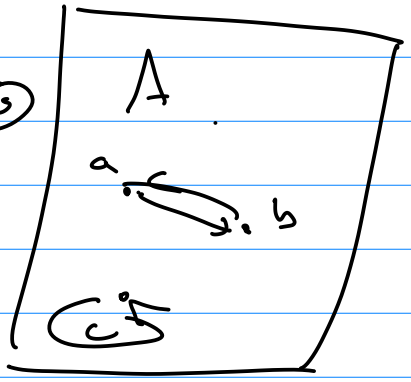
Note: $A \quad B$



$A \quad A$



(xs)



Properties of R, a relation on set A

① Reflexive $\forall a (aRa)$

② Irreflexive $\forall a (a \not R a)$

③ Symmetric $\forall a \forall b (aRb \to bRa)$

④ Antisymmetric $\forall a \forall b (a \neq b \to a \not R b \vee b \not R a)$

$\equiv \forall a \forall b (aRb \wedge bRa \to a = b)$

⑤ Assymetric $\forall a \forall b (aRb \to b \not R a)$

↳ irreflexive and antisym

⑥ Transitive $\forall a \forall b \forall c (aRb \wedge bRc \rightarrow aRc)$

ex: A is the set of $\mathbb{R}$

$R = \{(a,b) \mid a+b=0\}$

① Reflexive? $0+0=0$ — counter exampl.

No $1+1=2 \neq 0$ $1 \not R 1$

② Irreflexive? $1 R 1$? R ?

No $0 R 0$ ← counterexample

③ Symmetric? $\forall a \forall b (aRb \rightarrow bRa)$

$\forall a \forall b ((a+b=0) \rightarrow (b+a=0))$ Yes

④ Antisym? $\forall a \forall b (aRb \wedge bRa \rightarrow a=b)$

No $1+(-1)=0 \wedge (-1)+1=0 \rightarrow 1 \neq -1$

$1, -1$ ← counterexample

⑤ Asymmetry $\forall a \forall b (aRb \rightarrow bRa)$

No $\forall a \forall b (a+b=0 \rightarrow b+a \neq 0)$

counterexample $0+0=0$

⑥ Transitive $\forall a \forall b \forall c (aRb \wedge bRc \rightarrow aRc)$

$\forall a \forall b \forall c (a+b=0 \wedge b+c=0 \rightarrow a+c=0)$

No $a+b=0 \rightarrow b=-a$

$b+c = -a+c \stackrel{?}{=} 0$ $(1)+(-1)=0$, $(-1)+(1)=0$, but $1+1=2$ ↑ not zero!

Ops

① b/c R is a set → we can use set ops

Ex) $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 \oplus R_2$, $R_1 - R_2$, etc

② Composition $R \subseteq A \times B$ $S \subseteq B \times C$

Composition of S and R is a subset of $A \times C$

$$S \circ R = \{(a,c) \mid a \in A \wedge c \in C \wedge \exists b \in B \ni : aRb \wedge bRc\}$$

Note: (on order)

$$(f \circ g)(x) = f(g(x))$$

inside/out

③ R is on set A

Power: R^n, $n = 1, 2, 3, \ldots$

$$R^1 = R, \quad R^{n+1} = R^n \circ R$$

Thm $(R$ is transitive$) \iff (\forall n \ R^n \subseteq R)$ $n = 1, 2, \ldots$