Math 344

13.6 Surface S in $\mathbb{R}^3$ (x,y,z)

Explicit form $z = f(x,y)$

Implicit form $f(x,y,z) = 0$

parametric: $\vec{r}(u,v) = \langle x, y, z \rangle$

$x(u,v)$, $y(u,v)$, $z(u,v)$

on $(u,v) \in D$

parametrize domain.

Line Integral

$$\int_C \boxed{f(x,y,z)}\, ds = \int_a^b \boxed{f(\vec{r}(t))\, |\vec{r}\,'|}\, dt$$

curve in $\mathbb{R}^3$

along curve itself.

parametric form

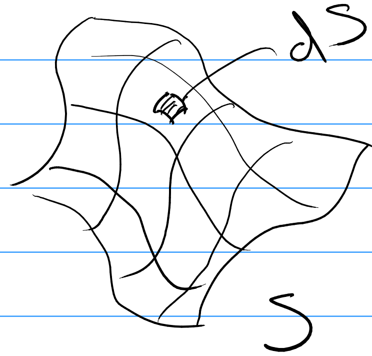
$\vec{r} = \langle x, y, z \rangle$

$x(t)$, $y(t)$, $z(t)$

$t \in [a,b]$

Surface Integrals

$\mathbb{R}^3$



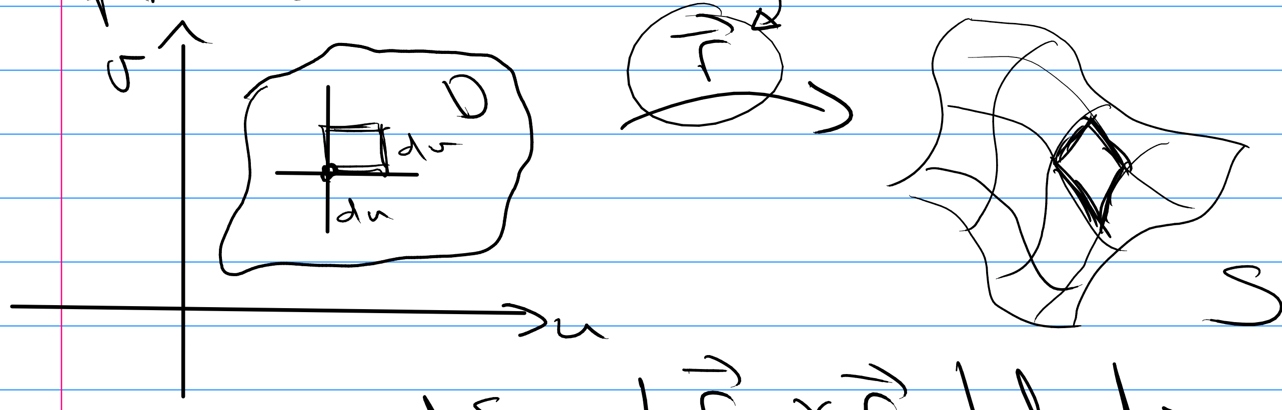
add up $\underbrace{f(x,y,z)}_{P \leftarrow \text{point}}$

$$\sum_j \sum_i f(P^*) \Delta S_{ij}$$

$$\iint_S f(x,y,z)\, dS = \lim_{\max \to 0} \sum_i \sum_j f(P^*_{ij}) \Delta S_{ij}$$

x,y,z $\mathbb{R}^3$ space

Parametric form of S



$$dS = |\vec{r}_u \times \vec{r}_v| \underbrace{du\, dv}_{dA}$$

in u,v $\mathbb{R}^2$ space

So $$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA$$

S in x,y,z $\mathbb{R}^3$ space

D in u,v parametric $\mathbb{R}^2$ space

Surface Integral of a Scalar Function $f(x,y,z)$ over surface S with parametric form

$$S:\ \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

on $(u,v) \in D$

Properties: S can be piecewise-smooth

$$S = S_1 \cup S_2 \cup \dots \cup S_n$$

all smooth

$$\iint_S f\,dS = \iint_{S_1} f\,dS + \iint_{S_2} f\,dS + \dots + \iint_{S_n} f\,dS$$

Applications

① $A(S) = ?$ let $f = 1$

$$\iint_S f(x,y,z)\,dS = \iint_S 1 \cdot dS$$

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v|\,dA$$ ← found in 13.6

② Centers of mass

S is a lamina in $\mathbb{R}^3$ with

$\rho(x,y,z)$ density

a) $$m = \iint_S \rho(x,y,z)\,dS$$

b) $$\bar{x} = \frac{1}{m}\iint_S x\rho\,dS$$

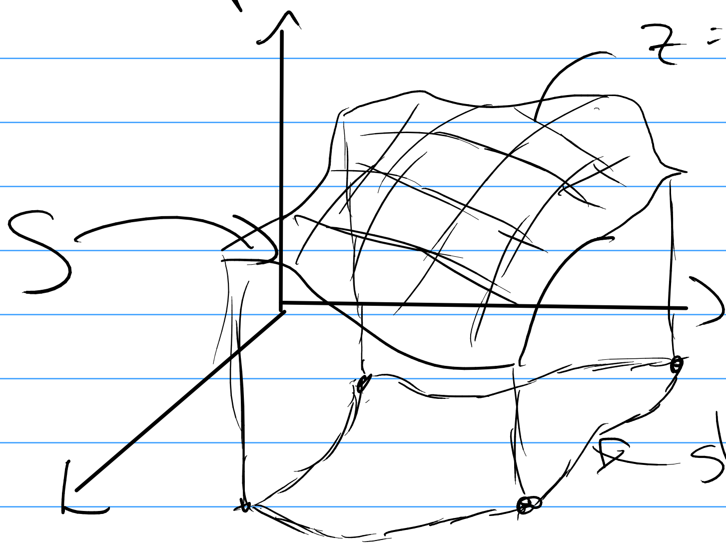
$$\bar{y} = \frac{1}{m}\iint_S y\rho\,dS$$

$$\bar{z} = \frac{1}{m}\iint_S z\rho\,dS$$

③ Explicit Functions

ex $z = f(x,y)$

trivial parametric form



$\vec{r} = \langle x, y, f(x,y) \rangle$

$(x,y) \in D$

$$\iint_S F(x,y,z)\, dS = \iint_D F(x,y,f(x,y))\, |\vec{r}_x \times \vec{r}_y|\, dA$$

$$|\vec{r}_x \times \vec{r}_y| = \left| \begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} \right|$$

$r = \langle x, y, f \rangle$

$r_x = \langle 1, 0, f_x \rangle$

$r_y = \langle 0, 1, f_y \rangle$

$$= |\langle -f_x, -f_y, 1 \rangle|$$

$$= \sqrt{f_x^2 + f_y^2 + 1}$$

$\boxed{\text{So}}$ $$\iint_S F\, dS = \iint_{D_{xy}} F(x,y,f(x,y)) \sqrt{f_x^2 + f_y^2 + 1}\, dA$$

ind. var

ex $\displaystyle\iint_S \left[(q+r^2)^{1/2}\, t\right] dS$

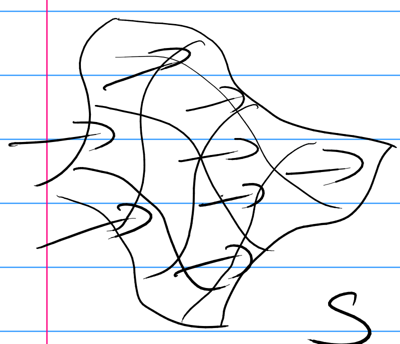
surface S is explicitly $q = \sin(r+t)$

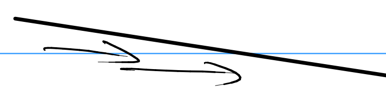
over $\boxed{0 \le r \le \pi \quad 0 \le t \le 2\pi} \leftarrow D$

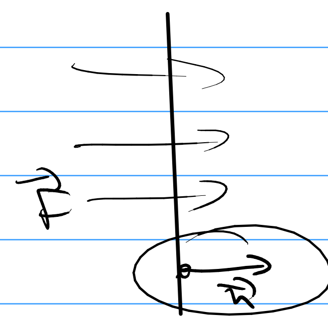
$$= \iint_D (\sin(r+t)+r^2)^{1/2} \cdot t \sqrt{q_r^2 + q_t^2 + 1}\, dA$$

$$= \int_0^{2\pi}\int_0^{\pi} (\sin(r+t)+r^2)^{1/2} \cdot t \cdot \sqrt{\cos^2(r+t) + \cos^2(r+t) + 1}\, dr\, dt$$

$=$ (you have to approx this one)







$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$ $\leftarrow$ pos. orientation

$-\vec{n}$ oppos. orientation

App (Vector Fields)

$$\iint_S \underbrace{(\vec{F} \cdot \vec{n})}_{\text{scalar function}} \, dS$$

$\boxed{ex}$ **flow of material.** $\vec{F} = \rho \cdot \vec{v}$

Velocity Vector field

Notation:

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iint_S \vec{F} \cdot d\vec{S}$$

$$d\vec{S} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \, dS$$