Math 243

~~Q's~~ Convergence $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}} = x + \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} + \dots$

$\boxed{\text{ratio test}}$ $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{|x|^{n+1}}{\sqrt{n+1}}}{\frac{|x|^n}{\sqrt{n}}}$

$\displaystyle = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} |x|$

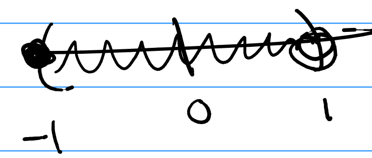
$\displaystyle = |x| \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = |x| \lim_{n\to\infty} \sqrt{\frac{1}{1+\cancel{1/n}^{\,0}}}$

$= |x| < 1$, conv

> 1 , div

or $= 1$, test fails

Conv. $|x| < 1$



$R = 1$

Interval (need endpoints still)

end pts

$x = -1$

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

$\displaystyle = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^{1/2}}$

Conv. alt. series

$x = 1$

$\displaystyle\sum_{n=1}^{\infty} \frac{(1)^n}{\sqrt{n}}$

$\displaystyle = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ p-series

that is divergent b/c

$\frac{1}{2} = p \leq 1$

change of index.

know: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = 1+2+3+\dots+n$

ex $\sum_{k=5}^{211} 3k+2 = (3\cdot 5+2)+(3\cdot 6+2)+\dots+(3\cdot 211+2)$

$i = k-4$

$$\sum_{i=1}^{207} 3(i+4)+2 = \sum_{i=1}^{207}(3i+14)$$

$K = i+4$

$$= 3\sum_{i=1}^{207} i + 14\sum_{i=1}^{207} 1$$

$$= 3\left[\frac{207(208)}{2}\right] + 14\left[207\right] = \boxed{?}$$

Q use Taylor Inequality..

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

$$f(x) = \underbrace{\boxed{f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n}}_{T_n(x)} + R_n(x)$$

$$f(x) = T_n(x) + R_n(x)$$

Taylor Inequality $|R_n(x)| \le \frac{M}{(n+1)!}(x-a)^{n+1}$

where $M \ge \left| f^{(n+1)}(x) \right|$ on $|x-a| \le d$

Ex $f(x) = x \cdot \boxed{\cos x}$ @ $a=0$ (Maclaurin)

Maclaurin of $\cos x$. $= 1 + 0 \cdot x - \frac{1}{2!}x^2 + 0x^3 + \frac{1}{4!}x^4 + \ldots$

$$\begin{aligned} f(x) &= \cos x & f(0) &= 1 \\ f'(x) &= -\sin x & f'(0) &= 0 \\ f''(x) &= -\cos x & f^{(2)}(0) &= -1 \\ f^{(3)}(x) &= \sin x & f^{(3)}(0) &= 0 \\ f^{(4)}(x) &= \cos x & f^{(4)}(0) &= 1 \\ &\vdots & &\vdots \end{aligned}$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \ldots$$

$$\boxed{x\cos x = x - \frac{1}{2!}x^3 + \frac{1}{4!}x^5 - \frac{1}{6!}x^7 + \ldots}$$

or $\cos(2x) = 1 - \frac{1}{2!}(2x)^2 + \frac{1}{4!}(2x)^4 - \frac{1}{6!}(2x)^6 + \ldots$

$$\boxed{\cos(2x) = 1 - \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 - \frac{2^6}{6!}x^6 + \ldots}$$

$@ x=0$

$f(x) = \cos(2x) \quad = \quad f(0) = 1$

$f'(x) = -2\sin(2x) \qquad f'(0) = 0$

$f^{(2)}(x) = -2^2\cos(2x) \qquad f^{(2)}(0) = -2^2$

$f^{(3)}(x) = 2^3\sin(2x) \qquad f^{(3)}(0) = 0$

$f^{(4)}(x) = 2^4\cos(2x) \qquad f^{(4)}(0) = 2^4$

$$\cos(2x) = 1 + 0x - \frac{2^2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{2^4}{4!}x^4 + \cdots$$

$$\boxed{\cos(2x) = 1 - \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 - \frac{2^6}{6!}x^6 + \cdots}$$

(15)

$@ x=0$

$f(x) = 0\sin x + x\cos x \qquad f(0) = 0$

$f'(x) = 1\cdot\cos x - x\sin x \qquad f'(0) = 1$

$f''(x) = -2\sin x - x\cos x \qquad f''(0) = 0$

$f^{(3)}(x) = -3\cos x + x\sin x \qquad f^{(3)}(0) = -3$

$f^{(4)}(x) = 4\sin x + x\cos x \qquad f^{(4)}(0) = 0$

$$x\cos x = 0 + 1\cdot x + \frac{0}{2!}x^2 + \frac{-3}{3!}x^3 + \cdots$$

$$\boxed{x\cos x = x - \frac{1}{2!}x^3 + \frac{1}{4!}x^5 - \dots}$$

Use Taylor Inequality

$$|R_n| \le \frac{M}{(n+1)!}|x-a|^{n+1} \qquad |f^{(n+1)}(x)| \le M \text{ on } |x-a| \le d$$

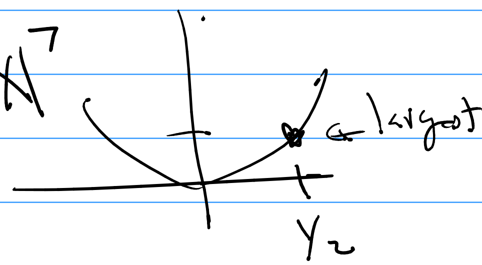
$$T_5 = T_6$$

$$x\cos x \approx x - \frac{1}{2!}x^3 + \frac{1}{4!}x^5$$

estimate

$$|R_7| \le \frac{M}{7!}|x|^7 \quad \text{on } |x| < 1/2$$

$$\text{so } |R_7| \le M\frac{(1/2)^7}{7!} \qquad |x|^7$$



M?

$$|f^{(7)}(x)| \le M \quad \text{on } |x| < \frac{1}{2}$$

$$f^{(7)}(x) = -7\cos x + x\sin x \le 7 + \frac{1}{2}$$

so let $M = 8$

$$|R_7| \le M \frac{(1/2)^7}{7!}$$

$$\rightarrow |R_7| \le \frac{8(1)^7}{7!} \approx 0.00001$$

on $|x| \le 1/2$

$$x\cos x \approx x - \frac{1}{2!}x^3 + \frac{1}{4!}x^5$$

error ≤ 0.00001

Exam 4 $\boxed{\text{Ch 11}}$ 12 probs. @ 10 pts $\rightarrow$ 110 pts = 100%

11.1 Seq's (1 prob)

a) given a seq $\rightarrow$ conv? div?
b)

Ex $\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}}$

$$= \sqrt{\lim_{n\to\infty} \frac{1}{1+\cancel{1/n}^{\,0}}} = \sqrt{\frac{1}{1}} = 1$$

ex $\left\{ \frac{1}{2}, \frac{-4}{3}, \frac{9}{4}, -\frac{16}{5}, \frac{25}{6}, \ldots \right\}$ ($n=1$, $n=2$, $n=3$, $n=4$)

$$a_n = \left\{ (-1)^{n+1} \frac{n^2}{n+1} \right\} \quad n = 1, 2, 3, \ldots$$

$$\lim_{n\to\infty} (-1)^{n+1} \frac{n^2}{n+1} \qquad \lim_{n\to\infty} \frac{n^2}{n+1} = \lim_{n\to\infty} \frac{1}{1/n + 1/n^2} = \infty$$

Div

ex $\lim_{n\to\infty} (-1)^{n+1} \frac{n+1}{n^2} = 0$ by squeeze thm

$$-\cancel{\frac{n+1}{n^2}} \to 0 \le \underline{(-1)^{n+1} \frac{n+1}{n^2}} \le \cancel{\frac{n+1}{n^2}} \to 0$$

b/c $\lim_{n\to\infty} \frac{n+1}{n^2} = \lim_{n\to\infty} \frac{1/n + 1/n^2}{1} = \frac{0}{1} = 0$

11.2 Series (Sum of seq) (1 prob)

① Find the sum of a convergent series

$\sum_{n=1}^{\infty} a_n$

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$S_n = a_1 + a_2 + \cdots + a_n$

$$\lim_{n\to\infty} S_n = ?_0$$

a) $S_n = a_1 + a_2 + \dots + a_n$ <u>telescoping</u>

$\boxed{ex}$ $S_n = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \cancel{\dots} + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$

$$S_n = 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \cancel{\frac{1}{n+1}}^{\,0}\right) = \boxed{1}$$

b) $S_n = 1 + \cancel{2/3} + \cancel{(2/3)^2} + \cancel{\dots} + \cancel{(2/3)^n}$

$\rightarrow \frac{2}{3} S_n = \cancel{2/3} + \cancel{(2/3)^2} + \cancel{(2/3)^3} + \cancel{\dots} + (2/3)^{n+1}$

$\boxed{Now}$ $S_n - \frac{2}{3} S_n = 1 - (2/3)^{n+1}$

$$\frac{1}{3} S_n = 1 - (2/3)^{n+1}$$

$$S_n = 3\left(1 - 2/3^{n+1}\right)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 3\left(1 - \cancel{(2/3)^{n+1}}^{\,0}\right) = \boxed{3}$$

$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 \dots$ <u>a div</u>

$S_1 = 1$

$S_2 = 1 - 1 = 0$

$S_3 = 1 - 1 + 1 = 1$

$S_4 = 1 - 1 + 1 - 1 = 0$

$S_n = \begin{matrix} 1 \\ \text{or} \\ 0 \end{matrix}$ $\lim_{n \to \infty} S_n$ d.n.e. $\boxed{\text{div.}}$

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5 problems

11.3 (Integral test, p-series)

11.4 (Comparison, limit comparison)

11.5 (Alt. series, Alt. series error estimates)

11.6 (Abs conv, ratio test, root test)

3 probs → I'll say what test to use

2 probs → you pick the test.

When is $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent? (use integral test)

Integral Test

① $\sum_{n=1}^{\infty} a_n$

② $f(x)$ has $f(n) = a_n$

all pos.

③ $\int_1^{\infty} f(x)\,dx$

(4) if integral div, then $\sum a_n$ div.
if integral conv, then $\sum a_n$ conv.

ex $\sum_{n=1}^{\infty} \frac{1}{n^p}$ let $f(x) = \frac{1}{x^p}$

$\to \int_1^{\infty} \frac{1}{x^p}\,dx = ?$

$\boxed{p=1}$ $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b\to\infty} \ln x \Big|_1^b$ $\boxed{\text{div}}$

$\boxed{p \neq 1}$ $\int_1^{\infty} x^{-p}\,dx = \lim_{b\to\infty} \frac{1}{1-p} x^{1-p} \Big|_1^b$ div on $p<1$ / conv. on $p>1$

<u>limit compare</u>

$\boxed{ex}$ $\sum_{n=1}^{\infty} \frac{(n+1)^3}{\sqrt{n^8+4n}}$ looks like $\sim \frac{n^3}{\sqrt{n^8}} = \frac{n^3}{n^4} = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{(n+1)^3}{\sqrt{n^8+4n}}}{1/n} = \lim_{n\to\infty} \frac{n(n+1)^3}{\sqrt{n^8+4n}} = \lim_{n\to\infty} \frac{n(n+1)^3/n^{4}}{\sqrt{n^8+4n}/n^4}$$

$$= \lim_{n\to\infty} \frac{(1+\cancel{1/n})^3}{\sqrt{1+\cancel{4/n^7}}} = \frac{1}{1} = 1 \text{ (pos. const.)}$$

$\therefore$ do the same thing.

$$\therefore \sum_{n=1}^{\infty} \frac{(n+1)^3}{\sqrt{n^8+4n}} \ \boxed{\text{div}}$$

$$\sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^3}{\sqrt{n^8+4n}}$$

alt. series

<u>check</u> $\lim_{n\to\infty} \frac{(n+1)^3}{\sqrt{n^8+4n}} = \lim_{n\to\infty} \frac{(1+\cancel{1/n})^3}{\sqrt{n^2+\cancel{4/n^5}}} = 0$

$\therefore$ $\boxed{\text{conv.}}$

$\boxed{\text{check for error}}$

ex) <u>w/ 2 terms</u> $S = \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^3}{\sqrt{n^8+4n}}$

$$\boxed{S \approx \frac{-8}{\sqrt{5}} + \frac{27}{\sqrt{264}}} \qquad |\text{error}| < \frac{4^3}{\sqrt{3^8+12}}$$

$\boxed{11.8}$ Power Series (1 prob)

① $\sum a_n$ $R = ?$

Interval of convergence?

$\boxed{Ex}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$

ratio test: $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\left(\frac{|2x-1|^{n+1}}{5^{n+1}\sqrt{n+1}}\right)}{\left(\frac{|2x-1|^n}{5^n\sqrt{n}}\right)}$

$\displaystyle = \lim_{n\to\infty} \frac{\cancel{5^n}}{5^{\cancel{n+1}}}\left(\frac{\sqrt{n}}{\sqrt{n+1}}\right)\frac{|2x-1|^{\cancel{n+1}}}{\cancel{|2x-1|^n}}$

$\displaystyle = \lim_{n\to\infty} \frac{1}{5}\cdot\sqrt{\frac{1}{1+\cancel{1/n}}}\cdot|2x-1|$

$\displaystyle = \frac{|2x-1|}{5}$ <1, conv

>1

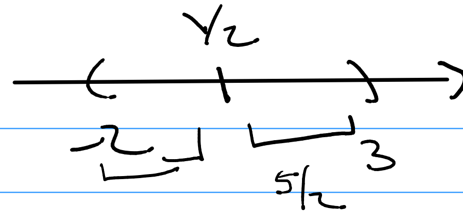
$=0$

So $\displaystyle\frac{|2x-1|}{5} < 1 \to |2x-1| < 5$

$\to -5 < 2x-1 < 5$

$\to -4 < 2x < 6$

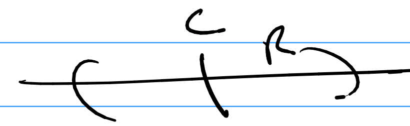
$\rightarrow \quad -2 < x < 3$



$$\boxed{R = 5/2}$$

d $\quad |2x-1| < 5$

1D

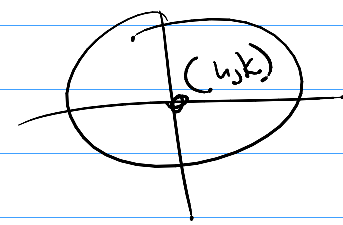


$2\,|x - 1/2| < 5$

$|x-c| < r$

$|x - 1/2| < 5/2$

2D



$(x-h)^2 + (y-k)^2 < r^2$

continue

$$\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \sqrt{n}}$$

$\boxed{\text{End points}}$

$x = -2$

$$\sum_{n=1}^{\infty} \frac{(-\cancel{5})^n}{\cancel{5^n}\sqrt{n}}$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$$

$\boxed{\text{conv.}}$ alt-series

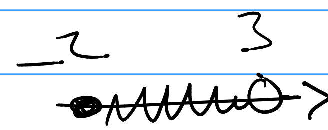
$x = 3$

$$\sum_{n=1}^{\infty} \frac{\cancel{5^n}}{\cancel{5^n}\sqrt{n}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

$\boxed{\text{div.}}$ p-series

$\therefore$ Interval $[-2, 3)$,



11.9 $\frac{1}{1-x} = 1 + x + x^2 + \cdots, \quad |x| < 1$

(1 prob)

"mess" with this to make new power series.

ex $\frac{1}{2+x^3} = \frac{1}{2}\left[\frac{1}{1-\left(-x^3/2\right)}\right]$

ex $\int \frac{1}{1+x}\,dx = \ln(1+x) = ?$

11.10 Taylor (Maclaurin Series).

(1 prob)

Know Table 1 p. 808

$\frac{1}{1-x} = ?$ $\quad \tan^{-1}x = ?$

$e^x = ?$ $\quad \ln(1+x) = ?$

$\sin x = ?$ $\quad (1+x)^k = ?$

$\cos x = ?$

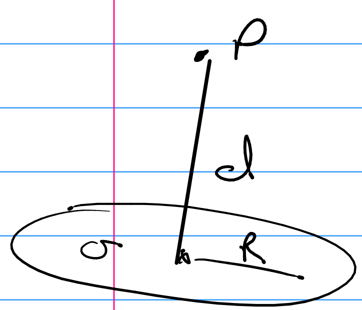
11.11 Apps (2 probs)

① Approx value $f(x) \approx T_n$

error $\leq$?

② Physics, given a function ≈ T_n

$R=1$ $\quad (1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$



$$V = \boxed{2\pi k \sigma}\left(\sqrt{d^2+R^2} - d\right)$$

as $d \gg R$ $\quad V \approx \frac{\pi k R^2 \sigma}{d}$

consider $(d^2+R^2)^{1/2} = d\left(1+\frac{R^2}{d^2}\right)^{1/2}$ ✓

or

$$\cancel{= R\left(1+\frac{d^2}{R^2}\right)^{1/2}}$$

$$\boxed{(d^2+R^2)^{1/2}} = d\left(1+\frac{R^2}{d^2}\right)^{1/2}$$

$$= d\left[1 + \frac{1}{2}\left(\frac{R^2}{d^2}\right) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(\frac{R^2}{d^2}\right)^2 + \dots\right]$$

$$= \boxed{d + \frac{d}{2}\left(\frac{R^2}{d^2}\right) + d\,\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{R^2}{d^2}\right)^2 + \dots}$$

Sub

$$V = 2\pi k \sigma\left(\sqrt{d^2+R^2} - d\right)$$

$$V = 2\pi k\sigma\left[\frac{1}{2}\frac{R^2}{d} + \boxed{\frac{d\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{R^2}{d^2}\right)^2} + \dots\right]$$

about zero

b/c $d >> R$ $\quad \frac{R}{d} \approx 0$

$$\therefore V \approx \frac{k\sigma \pi R^2}{d}$$