Math 243

Q's/ $\sum_{n=1}^{\infty} \frac{1}{6n-1} x^n = \frac{1}{5}x + \frac{1}{11}x^2 + \frac{1}{17}x^3 + \frac{1}{23}x^4 + \dots$

Radius / Interval of convergence

ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{1}{6n+5}|x|^{n+1}}{\frac{1}{6n-1}|x|^n} = \lim_{n\to\infty} \frac{6n-1}{6n+5}|x|$

$= |x|$ <1 conv. >1 div $=1$?

$|x|<1$; -1 0 1

endpoints? $x=-1$? $x=1$? conv? diverge?

a) $x=1$ $\sum_{n=1}^{\infty} \frac{(1)^n}{6n-1} = \sum_{n=1}^{\infty} \frac{1}{6n-1}$ looks like $\sum \frac{1}{n}$

use limit comparison test $\lim_{n\to\infty} \frac{\frac{1}{6n-1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{6n-1} = \frac{1}{6}$

pos. constant. ∴ act alike

b/c $\sum \frac{1}{n}$ div. $\to$ $\sum \frac{1}{6n-1}$ div.

b) $x=-1$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{6n-1} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{6n-1}$ alt. series.

b/c $\lim_{n\to\infty} \frac{1}{6n-1} = 0$ this is a conv. alt. series.

∴ $R=1$ and Interval $[-1, 1)$

Ex: $\displaystyle\sum_{n=2}^{\infty} \frac{1}{4(n!)} x^{n+3}$

ratio test: $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{1}{4(n+1)!}|x|^{n+4}}{\frac{1}{4\,n!}|x|^{n+3}}$

$\displaystyle= \lim_{n\to\infty} \frac{\cancel{n!}}{(n+1)\cancel{n!}} |x| = |x| \lim_{n\to\infty} \frac{1}{n} = |x|\cdot 0 = 0 < 1$ always true

So $R=\infty$, Interval $(-\infty,\infty)$

Note: $\displaystyle\sum_{n=1}^{\infty} \square$ ← something

by algebra & calculus

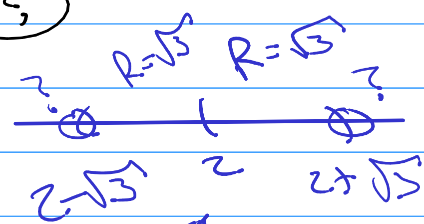
try ratio test: $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = (\sim) = \frac{|x-2|}{\sqrt{3}}$

So $\displaystyle\frac{|x-2|}{\sqrt{3}}$ <1 conv. >1 div. $=1$?

Conv: $\displaystyle\frac{|x-2|}{\sqrt{3}} < 1$

$|x-2| < \sqrt{3}$

$\boxed{R=\sqrt{3}}$



Interval?

a) if $x = 2-\sqrt{3}$ does $\sum \square$ → conv?

b) if $x = 2+\sqrt{3}$ does $\sum \square$ → div?

ex) approx $\int_0^{1/2} x^3 \tan^{-1} x \, dx$

know) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots, \quad R = 1$

So $x^3 \tan^{-1} x = x^4 - \frac{x^6}{3} + \frac{x^8}{5} - \frac{x^{10}}{7} + \dots, \quad R = 1$

$$\int x^3 \tan^{-1} x \, dx = \int \left(x^4 - \frac{x^6}{3} + \frac{x^8}{5} - \frac{x^{10}}{7} + \dots \right) dx$$

$$= \frac{1}{5} x^5 - \frac{1}{3 \cdot 7} x^7 + \frac{1}{5 \cdot 9} x^9 - \frac{1}{7 \cdot 11} x^{11} + \dots$$

$$\int_0^{1/2} x^3 \tan^{-1} x \, dx = \frac{1}{5}\left(\frac{1}{2}\right)^5 - \frac{1}{3 \cdot 7}\left(\frac{1}{2}\right)^7 + \frac{1}{5 \cdot 9}\left(\frac{1}{2}\right)^9 - \frac{1}{7 \cdot 11}\left(\frac{1}{2}\right)^{11} + \dots$$

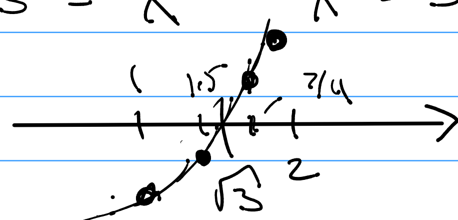
11.11) Applications (approximation of functions)

ex) $\sqrt{3}, \quad \sqrt[3]{9}, \quad \tan^{-1}(1.2)$

$\boxed{\sqrt{3} = ?}$

$\sqrt{3} = x \qquad x^2 = 3 \to \underline{x^2 - 3 = 0}$

① Bisection Method



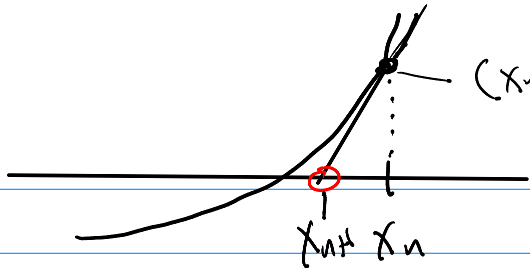
$1^2 - 3 = -2 \qquad 2^2 - 3 = 1$

$\left(\frac{3}{2}\right)^2 - 3 = \frac{9}{4} - 3 = \frac{9}{4} - \frac{12}{4} = -3/4$

$3(16)$

$\left(\frac{7}{4}\right)^2 - 3 = \frac{49}{16} - \frac{48}{16} = \frac{1}{16}$

② Newton's Method



$(x_n, f(x_n))$ $\quad m = f'(x_n)$

eq'n of line $\quad y - f(x_n) = f'(x_n)(x - x_n)$

$$0 - f(x_n) = f'(x_n)(x_{n+1} - x_n)$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

(3) $f(x) = x^{1/2}$ approx near $x = 4$

taylor expansion $\quad x^{1/2} = f(4) + f'(4)(x-4) + \frac{f''(4)}{2!}(x-4)^2 + \ldots$

$f(x) = \sqrt{x}$	$f(4) = 2$
$f'(x) = \frac{1}{2}x^{-1/2}$	$f'(4) = 1/4$
$f''(x) = -\frac{1}{4}x^{-3/2}$	$f''(4) = -1/32$
$f'''(x) = \frac{3}{8}x^{-5/2}$	$f^{(3)}(4) = 3/256$
$f^{(4)}(x) = -\frac{15}{16}x^{-7/2}$	$f^{(4)}(4) = -15/2048$

near $x = 4$ $\quad \sqrt{x} = \boxed{2 + \frac{1}{4}(x-4)} - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 - \ldots$

$$\sqrt{3} \approx 2 + \frac{1}{4}(3-4) = 7/4$$

$$\sqrt{3} \approx 2 + \frac{1}{4}(3-4) - \frac{1}{64}(3-4)^2 = 2 - 1/4 - \frac{1}{64}$$

$= 1.734375$

$\sqrt{3} \approx 1.73205080756888$

error

(1) alt. series? Yes $\rightarrow$ error < next term

(2) $|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$

when $|f^{(n+1)}(x)| \le M$ on $|x-a| < d$

Near $x=4 \to$ we use $\sqrt{3}$ so $\underset{3}{(}\overset{4}{|}\underset{5}{)}$

ex

$$\sqrt{x} = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 - \dots$$

$$f = \sqrt{x} \approx 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

$$|R_2| \le \frac{M}{3!}|x-4|^3 \le \frac{M}{3!}|1|^3 = \frac{M}{3!}$$

$$|f^{(3)}(x)| \le M \quad \text{on } (3, 5)$$

$$\frac{3}{8(x)^{5/2}} \le M \quad \text{on } (3, 5) \text{ with } 4$$

$$|f^{(3)}(x)| \le \frac{3}{8(3)^{5/2}} \le \boxed{\frac{3}{8} = M} \text{ at}$$

$$|R_2| \le \frac{M}{3!} \le \frac{3/8}{3!} = \frac{1}{16}$$

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ex $f(x) = \sinh(2x)$ approx near $x=0$ $R=1$

$n=5$ $\underline{\underline{T_5}}$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

$f(x) = \sinh(2x)$ $f(0) = 0$

$f'(x) = 2\cosh(2x)$ $f'(0) = 2(1)$

$f''(x) = 4\sinh(2x)$ $f''(0) = 0$

$f'''(x) = 8\cosh(2x)$ $f^{(3)}(0) = 8(1)$

$\vdots$

$$f^{(n)}(x) = \begin{cases} 2^n \sinh(2x), & n \text{ is even} \\ 2^n \cosh(2x), & n \text{ is odd} \end{cases} \qquad f^{(n)}(0) = \begin{cases} 0, & n \text{ is even} \\ 2^n, & n \text{ is odd} \end{cases}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh(2x) = 2x + \frac{2^3}{3!}x^3 + \frac{2^5}{5!}x^5 + \frac{2^7}{7!}x^7 + \dots$$

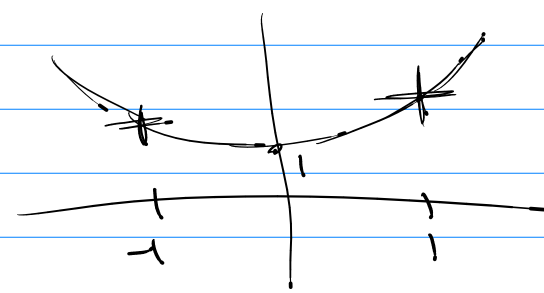
$$\boxed{\sinh(2x) \approx 2x + \frac{2^3}{3!}x^3 + \frac{2^5}{5!}x^5}$$ on $[-1, 1]$

error? $\quad |R_5| \le \frac{M}{6!}x^6$

where: $\quad |f^{(6)}(x)| \le M$ on $[-1, 1]$

$$|2^6 \cosh(2x)| \text{ on } [-1, 1]$$

$$\cosh(2x) = \frac{e^{4x} + e^{-4x}}{2}$$



$$\cosh(2x) \le \frac{e^4 + e^{-4}}{2} \le 28$$

so $|2^6 \cosh 2x| \le 2^6 \cdot 28 = \boxed{1792 = M}$

back to $\quad |R_5| \le \frac{M}{6!}|x|^6$ on $(-1, 1)$

$$\to \quad |R_5| \le \frac{1792}{6!}1^6 \approx 2.5$$

"Applied" Math = Physics

$$\boxed{K = \frac{1}{2} m v^2} \rightsquigarrow \text{Relativistic Physics.}$$

$c \approx 3 \times 10^6$ m/s ← speed of light $\qquad E = mc^2$

$$M = M_0 \left(\frac{1}{\sqrt{1-(v/c)^2}} \right)$$

$$\frac{1}{\sqrt{1-(v/c)^2}} \quad \text{let} \quad x = v/c$$

x: 0 → 1

$$(1-x^2)^{-1/2}$$

b/c $(1+\square)^k = 1 + k\square + \frac{k(k-1)}{2!}\square^2 + \frac{k(k-1)(k-2)}{3!}\square^3 + \dots$

$$(1-x^2)^{-1/2} = 1 + \frac{1}{2}x^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}x^4 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}x^6 + \dots$$

$$= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \dots$$

$$M = M_0 \left(1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \dots \right)$$

$x = v/c \qquad v = 0,\ x = 0$

Not moving? $\underline{m = M_0}$

$x \to 1$? $\qquad M \to \infty$

Energy

$$E = mc^2$$

@ rest $E_0 = m_0 c^2$

Kinetic Energy $K = E - E_0$ (low speed $K = \frac{1}{2}mv^2$)

$$K = mc^2 - m_0c^2 = c^2(m - m_0)$$

b/c $m = m_0\left(1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \dots\right)$

$$m = m_0 + m_0\left(\frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \dots\right)$$

$$m - m_0 = m_0\left(\frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \dots\right)$$

So $K = c^2 m_0\left(\frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \dots\right)$

$$K = \frac{1}{2}m_0c^2x^2 + \underbrace{c^2 m_0}_{E_0}\left(\frac{3}{8}x^4 + \frac{5}{16}x^6 + \dots\right)$$

b/c $x = v/c$ $\quad x^2 = v^2/c^2$

$$K = \boxed{\frac{1}{2}m_0v^2} + E_0\left(\frac{3}{8}x^4 + \frac{5}{16}x^6 + \dots\right)$$

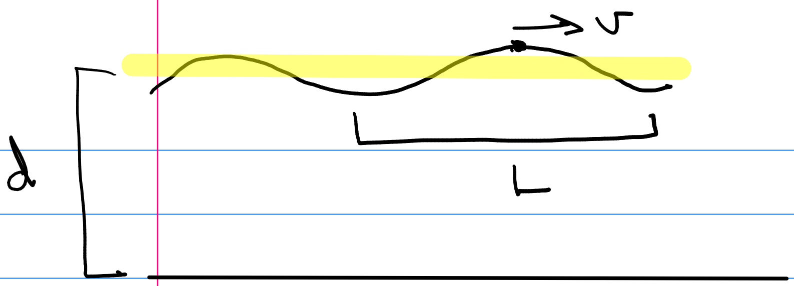
if $v << c$

$x = \frac{v}{c} \approx 0$

$c \approx 3{,}000{,}000$ m/s

$$v = 50\,\frac{mi}{hr} = 50\left(\frac{1\,hr}{3600\,s}\right)\left(\frac{1609}{1\,mi}\right) \approx 22\ m/s$$

$x = \frac{v}{c} = 0.0000008 \quad x^4 \approx 1\times10^{-21}$



$$v^2 = \frac{gL}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)$$

$$\tanh x = \frac{\sinh x}{\cosh x} \quad \begin{matrix} \sinh x \to x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \\ \cosh x \to 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \end{matrix}$$

$$x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \cdots$$

$$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \;\Big|\; x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$x + \frac{x^3}{2!} + \frac{x^5}{4!} + \frac{x^7}{6!} + \cdots$$

$$\left(\frac{1}{3!} - \frac{1}{2!}\right)x^3 + \left(\frac{1}{5!} - \frac{1}{4!}\right)x^5 + \cdots$$

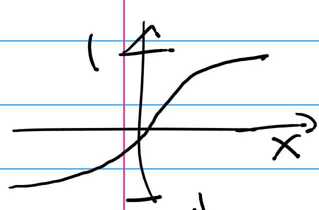
$$-\tfrac{1}{3}x^3 + \cdots$$

$$0 + \cdot$$

$$\tanh(x) = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \cdots$$

$$v^2 = \frac{gL}{2\pi}\tanh\left(\frac{2\pi d}{L}\right)$$

<u>Note:</u> (1) <u>deep water.</u> $d >> L \to \frac{d}{L} \approx \infty$



$$v^2 \sim \lim_{d\to\infty} \frac{gL}{2\pi}\,\cancel{\tanh}\overset{1}{\left(2\pi\frac{d}{L}\right)} = \frac{gL}{2\pi}$$

$$\boxed{V \approx \sqrt{\frac{gL}{2\pi}}}$$

② Shallow $\frac{d}{L}$ $d < L$ $\frac{d}{L} \to 0$
(near zero)

$$\tanh(x) = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \dots$$

$$v^2 = \frac{gL}{2\pi}\tanh\left(2\pi \frac{d}{L}\right)$$

take only T_1

$$v^2 \approx \frac{g\cancel{L}}{\cancel{2\pi}}\left(\cancel{2\pi}\frac{d}{\cancel{L}}\right)$$

$$v^2 \approx gd \to \boxed{v \sim \sqrt{gd}}$$

Deep: $v \approx \sqrt{\frac{gL}{2\pi}}$

Shallow $v \approx \sqrt{gd}$

$$\frac{\frac{\cancel{g}L}{2\pi}}{\cancel{g}d} = \frac{L}{2\pi d} > 1$$

or

< 1