Math 243

~~Q's/~~

Taylor / Maclaurin series

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

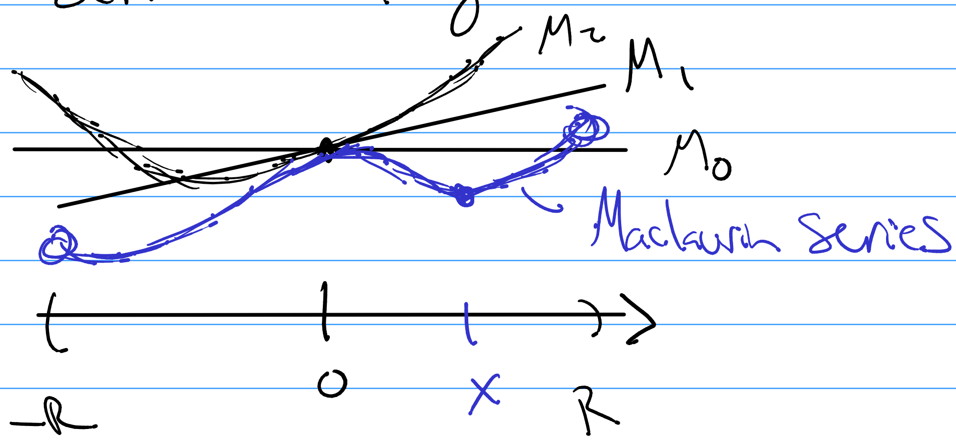
Polynomials:

$M_0 = f(0)$

$M_1 = f(0) + f'(0)x = M_0 + f'(0)x$

$M_2 = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = M_1 + \frac{f''(0)}{2!}x^2$

If series converges on an interval $(-R, R)$



Does: given $f(x) \overset{?}{=}$ Taylor / Maclauren series on its interval of convergence

→ When are they equal?

→ how to use the series for calculations?

For our problems:

1. generate Taylor/Maclaurin series
2. find intervals of convergence
3. show $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$

   on interval of convergence

(Make table #1 p. 808)

4. use the series to "do stuff."

   a) make new series
   b) actual number calculations with error
   c) definite integrals using series.

$\boxed{\text{ex}}$ $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad R = \infty$

$F(x) = \int \sin(x^2)\,dx$

$F(x) = \int \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) dx$

$= C + \frac{1}{2}x^2 - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \dots$

$$\rightarrow \boxed{F(x) \approx \frac{x^2}{2!} - \frac{x^4}{4!}} \quad \text{given } C=0$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, \quad R = \infty$$

use $\lim_{n \to \infty} |R_n| = 0$ when?

$$\boxed{\text{use}} \quad e = e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$$

$\frac{1}{2}\left(\frac{1}{1!}\right)$ $\quad \frac{1}{3}\left(\frac{1}{2!}\right)$ $\quad \frac{1}{4}\left(\frac{1}{3!}\right)$

calculate: (partial sums, partial products)

$a = 1$
$p = 1$
for $i = 1$ to 100
$\quad p = \left(\frac{1}{i} \cdot p\right)$
$\quad a = a + p$
end

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots, \quad R = \infty$$

Maclaurin Series (expanding at zero)

$\rightarrow$ converges fastest near zero.

$\to \boxed{e^x}$ $e^{101} \approx ?$ (101 is far from zero) need $\boxed{\text{lots}}$ of terms

$e^x = 1 + x + \frac{x^2}{2!} + \dots$

$e^{101} = 1 + (101) + \frac{(101)^2}{2!} + \frac{(101)^3}{3!} + \dots$

"cheat" $\left(e^{x/n}\right)^n = e^x$

call near zero to be smaller than 1/2

$$e^{101} = \left(e^{\frac{101}{202}}\right)^{202} = \left(e^{1/2}\right)^{202}$$

$\to$ find $e^{0.5} = 1 + .5 + \frac{(.5)^2}{2!} + \frac{(.5)^3}{3!} + \dots$

$\to e^{101} = \left(e^{0.5}\right)^{202}$

$\sin(x) = ?$

Maclaurin @ $x=0$

$f(x) = \sin x$	$f(0) = 0$
$f'(x) = \cos x$	$f'(0) = 1$
$f''(x) = -\sin x$	$f''(0) = 0$
$f'''(x) = -\cos x$	$f'''(0) = -1$
$f^{(4)}(x) = \sin x$	$f^{(4)}(0) = 0$

Maclaurin Coeffs. $0, \frac{1}{1!}, \frac{0}{2!}, \frac{-1}{3!}, \frac{0}{4!}, \frac{1}{5!}, \frac{0}{6!}, \frac{-1}{7!}$

Maclaurin Series:

$$0 + x + 0x^2 - \frac{1}{3!}x^3 + 0x^4 + \frac{1}{5!}x^5 + 0x^6 - \frac{1}{7!}x^7 + \dots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

alternating series

converges when

$$\lim_{k\to\infty} \frac{x^{2k+1}}{(2k+1)!} = 0 \quad \left(\begin{array}{c}\text{always}\\ \text{true}\end{array}\right)$$

$$R = \infty$$

So $\boxed{\text{is}}$ $\sin(x) \overset{?}{=} x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

use Taylor inequality: if $\left| f^{(n+1)}(x) \right| \le M$ for $|x-a| \le d$

then $|R_n| \le \frac{M}{(n+1)!} |x-a|^{n+1}$ for $|x-a| \le d$

but $f^{(k)}(x) = \begin{array}{c} \pm \sin x \\ \pm \cos x \end{array} \le 1$ for all x

So $|R_n| \le \frac{1}{(n+1)!}|x|^{n+1}$

$$\lim_{n\to\infty} |R_n| \le \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

So $\lim_{n\to\infty} |R_n| = 0$

Yep $\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, \quad R = \infty}$

Use it?

$$\sin(0.2) = 0.2 - \frac{(0.2)^3}{3!} + \frac{(0.2)^5}{5!} - \dots$$

$$\sin(0.2) \approx 0.2 - \frac{(0.2)^3}{3!} \approx 0.19867$$

$$\text{error} \le \frac{0.2^5}{5!} \approx 0.0000027$$

$$\sin(101) = 101 - \frac{101^3}{3!} + \frac{101^5}{5!} - \frac{101^7}{7!} + \dots$$

goal: error $\le 0.1 \to \frac{101^n}{n!} \le 0.1$

$\rightarrow \boxed{101^n \leq 0.1 \cdot n!}$ alot! $n \approx 301$

$$\boxed{n \leq \frac{\ln(0.1\, n!)}{\ln(101)}}$$

$$\sin(x + 2\pi n) = \sin(x)$$

$$\sin(\underbrace{\bmod(101, 2\pi)}_{\text{maps 101 to between } [0, 2\pi]})$$

$\boxed{\bmod}$ division_algorithm

$$7 = 2 \cdot \boxed{3} + 1$$

quotient (div) remainder (mod)

$$101 = q \cdot (2\pi) + \boxed{r}$$

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ex Binomial th^m $(n = 0, 1, 2, 3, \ldots)$

$(a+b)^n$

Pascal's triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

$(a+b)^0 = 1$

$(a+b)^1 = a+b$

$(a+b)^2 = a^2 + 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$$(a+b)^n = \frac{n!}{n!\,0!}a^n b^0 + \frac{n!}{(n-1)!\,1!}a^{n-1}b^1 + \frac{n!}{(n-2)!\,2!}a^{n-2}b^2 + \cdots + \frac{n!}{0!\,n!}a^0 b^n$$

choose: n choose k $= \binom{n}{k} = \dfrac{n!}{(n-k)!\,k!}$

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}a^0 b^n$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

ex $(a+b)^{100} = \binom{100}{0}a^{100}b^0 + \binom{100}{1}a^{99}b + \cdots + \binom{100}{100}a^0 b^{100}$

$= a^{100} + 100a^{99}b + \cdots + b^{100}$

$$(1+x)^k = \sum_{i=0}^{k}\binom{k}{i}1^{k-i}x^i = \sum_{i=0}^{k}\binom{k}{i}x^i$$

if $x = 0, 1, 2, \ldots$

$$(1+x)^k = 1 + \binom{k}{1}x + \binom{k}{2}x^2 + \binom{k}{3}x^3 + \cdots + \binom{k}{k}x^k$$

$$= 1 + kx + \frac{k!}{(k-2)!\,2!}x^2 + \frac{k!}{(k-3)!\,3!}x^3 + \cdots + x^k$$

$$(1+x)^k = 1 + \frac{kx}{1!} + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots + x^k$$

Consider

Maclaurin?

$f(x) = (1+x)^{k}$ @ $x = 0$

$f(0) = 1$

$f'(x) = k(1+x)^{k-1}$ $\quad f'(0) = k$

$f''(x) = k(k-1)(1+x)^{k-2}$ $\quad f^{(2)}(0) = k(k-1)$

$f'''(x) = k(k-1)(k-2)(1+x)^{k-3}$ $\quad f^{(3)}(0) = k(k-1)(k-2)$

$$f^{(n)}(0) = k(k-1)(k-2)\cdots(k-(n-1))$$
$$= k(k-1)\cdots(k-n+1)$$

Maclaurin series

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

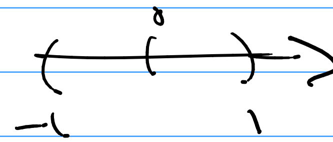
$$= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots + \frac{k(k-1)\cdots(k-n+1)}{n!}x^n + \cdots$$

Converge?

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{\cancel{k(k-1)(k-2)\cdots(k-n+1)}(k-n)\,|x|^{\cancel{n}+1}}{(n+1)\cancel{n!}}}{\frac{\cancel{k(k-1)(k-2)\cdots(k-n+1)}\,\cancel{|x|^n}}{\cancel{n!}}}$$

$$\lim_{n\to\infty} \frac{|k-n|}{n+1}|x| = |x| \begin{matrix} <1 & \text{means convergence} \\ >1 & \text{means divergence} \end{matrix}$$

Know: $1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$, $R = 1$



$\boxed{\text{Interval?}}$ $x = 1$, $x = -1$

@ $x = \pm 1$ it conv. if $k \geq 0$

@ $x = 1$ it conv. if $-1 < k \leq 0$

$$\sqrt{1+x}$$

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3 + \cdots$$

on $\underline{\underline{R=1}}$

Use?

$$\int_0^{.4} \sqrt{1+x^4}\,dx \qquad |\text{error}| < 0.000005$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots, \quad R = 1$$

$$(1+x^4)^{1/2} = 1 + \frac{1}{2}x^4 + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^8 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^{12} + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{4!}x^{16} + \cdots$$

$$(1+x^4)^{1/2} = 1 + \frac{1}{2}x^4 + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^8 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}x^{12} + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!}x^{16} + \cdots$$

$$(1+x^4)^{1/2} = 1 + \frac{1}{2}x^4 - \frac{1}{2^2 \cdot 2!}x^8 + \frac{3\cdot 1}{2^3\, 3!}x^{12} - \frac{5\cdot 3\cdot 1}{2^4\, 4!}x^{16} + \cdots$$

$$\int (1+x^4)^{1/2}\,dx$$

$$= C + x + \frac{1}{2}\cdot\frac{x^5}{5} - \frac{1}{2^2\, 2!}\frac{x^9}{9} + \frac{3\cdot 1}{2^3\cdot 3!}\frac{x^{13}}{13} - \frac{5\cdot 3\cdot 1}{2^4\cdot 4!}\frac{x^{17}}{17} + \cdots$$

$$\int_0^{.4} (1+x^4)^{1/2}\,dx = \boxed{(.4) + \frac{1}{2}\frac{(.4)^5}{5}} - \underbrace{\frac{1}{2^2\, 2!}\frac{(.4)^9}{9}}_{\leq\, 0.000004} + \frac{3\cdot 1}{2^3\, 3!}\frac{(.4)^{13}}{13} - \cdots$$

$$\boxed{(.4) + \frac{1}{2}\frac{(.4)^5}{5}} = ?$$

Series $\quad e^x \sin x = \sum_{k=0}^{\infty} ?$

<u>use distributive:</u> **ex** $\quad (a+b)(c+d) = ac + ad + bc + bd$

So $\quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\Rightarrow e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$e^x \sin x = x + x^2 + \left(\frac{1}{2!} - \frac{1}{3!}\right)x^3 + \left(\frac{1}{3!} - \frac{1}{3!}\right)x^4 + \cdots$$