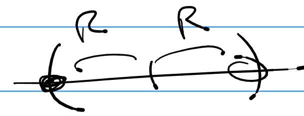
Math 243

Q5f $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$, on $|x-a| < R$



$\boxed{\text{app}}$ $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ on $\underline{|x|<1}$ domain

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$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$, on $|x-a| < R$

$f'(x) = 0 + c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$, on $|x-a| < R$

$\int f(x)\,dx = c_0 x + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \dots$, on $|x-a| < R$

$\boxed{\text{So}}$ $\frac{1}{1-\square} = 1 + \square + \square^2 + \dots$, on $|\square| < 1$

New power series by .. ① Algebra / Substitution
② Calculus

$\boxed{\text{ex}}$ $\frac{d}{dx}\left[\tan^{-1} x\right] = \frac{1}{1+x^2}$ $\qquad$ $\frac{d}{dx}\left\{\tanh^{-1} x\right\} = \frac{1}{1-x^2}$

$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x$ $\qquad$ $\int \frac{1}{1-x^2}\,dx = \tanh^{-1} x$

$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \dots$, on $|-x^2| < 1$

$$\boxed{\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \dots, \text{ on } |x|<1}$$

$$\frac{1}{1+(1/3)^2} \approx 1 - (1/3)^2 + (1/3)^4 - (1/3)^6$$

error $< (1/3)^8$ b/c alt. series

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but $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x$

$\Rightarrow \tan^{-1} x = \int \frac{1}{1+x^2}\,dx$ on $|x|<1$

$\tan^{-1} x = \int (1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \dots)\,dx$

$\tan^{-1} x = C + x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$ on $|x|<1$

but $\tan^{-1}(0) = 0 \Rightarrow 0 = C$

$$\boxed{\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots, \text{ on } |x| \leq 1}$$

ex $\tan^{-1}(.12) = (.12) - \frac{1}{3}(.12)^3 + \frac{1}{5}(.12)^5 - \dots$

$\theta = \tan^{-1}(.12)$ $\qquad \pm \frac{1^\circ}{2} \cdot \frac{\pi}{180^\circ} = \pm \frac{\pi}{360} \approx \boxed{\pm 0.001}$ error

$\theta = .12 - \frac{1}{3}(.12)^3 + \frac{1}{5}(.12)^5 - \frac{1}{7}(.12)^7 + \frac{1}{9}(.12)^9 - \dots$

$\theta \approx .12 - \frac{1}{3}(.12)^3$, error $< \frac{1}{5}(.12)^5 \approx 0.000005$

$\Rightarrow \theta \approx 0.119424 \approx 0.12$ ~~radians~~ $\left(\frac{180^\circ}{\pi}\right) \approx \boxed{6.9^\circ}$

$\tan^{-1}(0) = 0$

$\tan^{-1}(1) = \pi/4$

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots}$$ Leibniz formula for π

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots\right)$$

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \frac{4}{13} - \frac{4}{15} + \dots$$

$$\pi \approx 4 - \frac{4}{3} = \frac{8}{3}, \quad \text{error} < \frac{4}{5}$$

$$\pi \approx 4 - \frac{4}{3} + \frac{4}{5} = \frac{52}{15}, \quad \text{error} < \frac{4}{7}$$

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ex $\tanh^{-1}(x) = \int \frac{1}{1-x^2}\,dx$

Know $\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \dots$, on $|x^2| < 1$

$$\tanh^{-1}(x) = \int (1 + x^2 + x^4 + x^6 + \dots)\,dx, \text{ on } |x| < 1$$

$$\tanh^{-1}(x) = C + x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \dots, \text{ on } |x| < 1$$

$\tanh^{-1}(0) = C$ → $\tanh(y) \stackrel{?}{=} 0$ $\quad \frac{\sinh(y)}{\cosh(y)} \stackrel{?}{=} 0$

→ $\sinh y = 0$ → $\frac{e^y - e^{-y}}{2} = 0$ $\quad \boxed{y = 0}$

So $C = 0$

$$\boxed{\tanh^{-1}(x) = x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 + \dots, \text{ on } |x|<1}$$

Ex) $f(x) = \dfrac{x^2+x}{(1-x)^3}$ Know: $\dfrac{1}{1-x} = 1+x+x^2+x^3+\dots$

$\Rightarrow f(x) = \dfrac{x^2}{(1-x)^3} + \dfrac{x}{(1-x)^3}$

$\dfrac{d}{dx}\left[(1-x)^{-1}\right] = \dfrac{d}{dx}\left[1+x+x^2+x^3+\dots\right]$

$(-1)(1-x)^{-2}(-1) = 1+2x+3x^2+\dots$

$\dfrac{1}{(1-x)^2} = 1+2x+3x^2+\dots$

$\Rightarrow f(x) = \dfrac{x^2}{2}\left(\dfrac{2}{(1-x)^3}\right) + \dfrac{x}{2}\left(\dfrac{2}{(1-x)^3}\right)$

$\dfrac{d}{dx}\left[(1-x)^{-2}\right] = \dfrac{d}{dx}\left[1+2x+3x^2+4x^3+\dots\right]$

$\dfrac{2}{(1-x)^3} = 2+3\cdot2x+4\cdot3x^2+5\cdot4x^3+\dots$

So $f(x) = \dfrac{x^2}{2}\left[2+3\cdot2x+4\cdot3x^2+5\cdot4x^3+\dots\right]$

$+ \dfrac{x}{2}\left[2+3\cdot2x+4\cdot3x^2+5\cdot4x^3+\dots\right]$

$f(x) = \left[x^2 + \dfrac{3\cdot2}{2}x^3 + \dfrac{4\cdot3}{2}x^4 + \dfrac{5\cdot4}{2}x^5 + \dots\right]$

$+ \left[x + \dfrac{3\cdot2}{2}x^2 + \dfrac{4\cdot3}{2}x^3 + \dfrac{5\cdot4}{2}x^4 + \dots\right]$

$f(x) = \left(\dfrac{1\cdot0}{2}+\dfrac{2\cdot1}{2}\right)x^1 + \left(\dfrac{2\cdot1}{2}+\dfrac{3\cdot2}{2}\right)x^2 + \left(\dfrac{3\cdot2}{2}+\dfrac{4\cdot3}{2}\right)x^3 + \left(\dfrac{4\cdot3}{2}+\dfrac{5\cdot4}{2}\right)x^4 +$

$+\left(\dfrac{5\cdot4}{2}+\dfrac{6\cdot5}{2}\right)x^5 + \left(\dfrac{\boxed{6\cdot5}}{2}+\dfrac{\boxed{7\cdot6}}{2}\right)x^6 + \dots$ on $|x|<1$

So $f(x) = \displaystyle\sum_{k=1}^{\infty} k^2 x^k$, on $|x|<1$

$$\therefore \quad \frac{x^2+x}{(1-x)^3} = x + 4x^2 + 9x^3 + 16x^4 + 25x^5 + \dots, \text{ on } |x|<1$$

Approximate definite integrals.

$\boxed{\text{ex}}$ $\displaystyle\int_0^{1/2} \tan^{-1}(x/2)\,dx$

b/c $\tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$, on $|x| \le 1$

$\tan^{-1}(x/2) = \frac{x}{2} - \frac{1}{3\cdot 2^3}x^3 + \frac{1}{5\cdot 2^5}x^5 - \frac{1}{7\cdot 2^7}x^7 + \dots$, on $|x| \le 2$

$$\int_0^{1/2}\left[\frac{x}{2} - \frac{1}{3\cdot 2^3}x^3 + \frac{1}{5\cdot 2^5}x^5 - \frac{1}{7\cdot 2^7}x^7 + \dots\right]dx$$

$$= \frac{x^2}{2\cdot 2} - \frac{1}{4\cdot 3\cdot 2^3}x^4 + \frac{1}{6\cdot 5\cdot 2^5}x^6 - \frac{1}{8\cdot 7\cdot 2^7}x^8 + \dots \Big|_0^{1/2}$$

$\boxed{\text{Note}}$ $\displaystyle\sum_{K=2,4,6,\dots} \frac{(-1)^{K/2+1}}{K(K-1)2^{K-1}} x^K$, $K=2,4,6,\dots$

$$= \left[\frac{1}{2\cdot 2}\left(\frac{1}{2}\right)^2 - \frac{1}{4\cdot 3\cdot 2^3}\left(\frac{1}{2}\right)^4 + \frac{1}{6\cdot 5\cdot 2^5}\left(\frac{1}{2}\right)^6 - \dots\right] - [0]$$

$\displaystyle\int_0^{1/2} \tan^{-1}(x/2)\,dx \approx \frac{1}{16} - \frac{1}{3\cdot 2^9}$, error $< \frac{1}{3\cdot 5\cdot 2^{12}}$

$\approx \boxed{0.06184}$, error < 0.00002

$\boxed{11.10}$ Taylor / Maclaurin Series

Consider: $\boxed{f(x)} = \boxed{c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots}$, on $|x-a|<R$

$f'(x) = \boxed{c_1} + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots$

$f''(x) = \boxed{2\cdot 1\, c_2} + 3\cdot 2\, c_3(x-a) + 4\cdot 3\, c_4(x-a)^2 + \ldots$

$f'''(x) = \boxed{3\cdot 2\cdot 1\, c_3} + 4\cdot 3\cdot 2\, c_4(x-a) + 5\cdot 4\cdot 3\cdot c_5(x-a)^2 + \ldots$

$\vdots$

$f^{(n)}(x) = \boxed{n!\, c_n} + \boxed{(n+1)(n)(n-1)\cdots(2)}\, c_{n+1}(x-a) + \ldots$

So

$$\begin{aligned}
f(a) &= 0!\, c_0 + 0 \rightarrow c_0 = f(a)\\
f'(a) &= 1!\, c_1 + 0 \rightarrow c_1 = f'(a)\\
f''(a) &= 2!\, c_2 + 0 \rightarrow c_2 = \frac{1}{2!} f''(a)\\
&\vdots\\
f^{(n)}(a) &= n!\, c_n + 0 \rightarrow c_n = \frac{1}{n!} f^{(n)}(a)
\end{aligned}$$

$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$, on $\boxed{|x-a|<R}$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

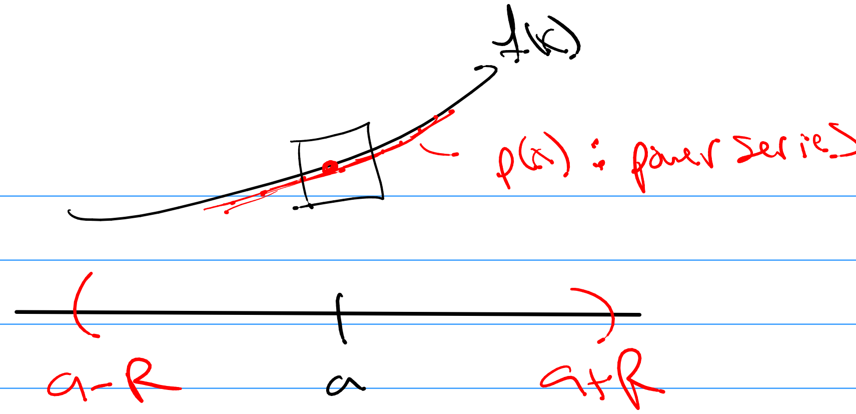
"Check" $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots$ on $|x|<1$

$f(x) = \frac{1}{1-x}$ $\quad f'''(x) = \frac{3\cdot 2\cdot 1}{(1-x)^4}$ $\quad c_0 = f(0) = 1$

$f'(x) = \frac{1}{(1-x)^2}$ $\quad c_1 = f'(0) = 1$

$f''(x) = \frac{2\cdot 1}{(1-x)^3}$ $\quad f^{4}(x) = \frac{4\cdot 3\cdot 2\cdot 1}{(1-x)^5}$ $\quad c_2 = \frac{f''(0)}{2!} = \frac{2!}{2!} = 1$

graph idea @ $x=a$



$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

taylor series of $f(x)$ about $x=a$

let $a=0$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

Maclaurin Series.

Warning: the above is under the assumption that

$f(x)$ = power series on $|x-a| < R$

goal

$e^x = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots$, on $|x| < R$

if this did exist

Maclaurin $e^x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$

$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$, on $|x| < R$

But does e^x have a power series?

Better yet... when does any $f(x)$ have a power series?

Taylor Series

$$f(x) \stackrel{?}{=} f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

Need: $f(x)$ must be infinitely differentiable @ $x=a$.

Partial sums

$T_0(x) = f(a)$

$T_1(x) = f(a) + f'(a)(x-a)$

$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$ (n-th degree taylor polynomial)

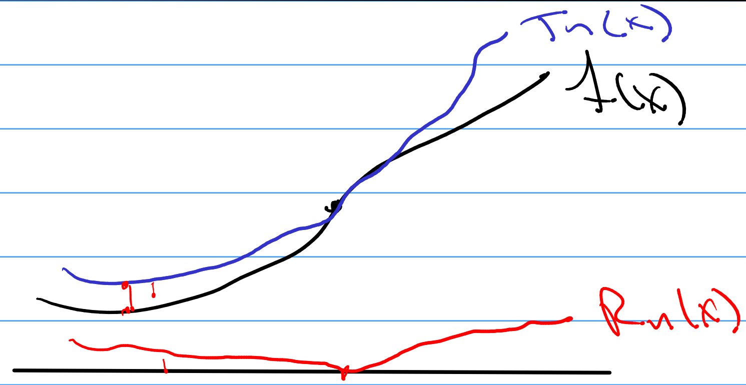
So $f(x) = \boxed{\lim_{n\to\infty} T_n(x)}$

New idea: can we do $\lim_{n\to\infty} T_n(x)$

$f(x)$, $T_n(x) = f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

Let $R_n(x) = f(x) - T_n(x)$

$\left(\text{so } f(x) = T_n(x) + R_n(x) \right)$



New Idea.

$$\lim_{n\to\infty} R_n(x) = 0$$

Thm if $f(x) = T_n(x) + R_n(x)$

T_n is the n^{th} degree Taylor polynomial of f @ $x=a$

and $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$

Then $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$ on $|x-a| < R$

Thm Taylor's Inequality

if $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$

then $|R_n(x)| \le \boxed{\frac{M}{(n+1)!}|x-a|^{n+1}}$ for $|x-a| \le d$

$$f(x) = \overbrace{\underbrace{f(a) + f'(a)(x-a) + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n}}^{T_n} + \overbrace{\underbrace{\frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \dots}}^{R_n}$$

<u>**Note:**</u> <u>**Know:**</u> $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for all x.

Prove $\quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$ on $(-\infty, \infty)$ $\quad R = \infty$

<u>b/c</u> $\frac{d^n}{dx^n}[e^x] = e^x$

So $|R^n| \le \frac{e^d}{(n+1)!}|x|^{n+1}$

$$\lim_{n\to\infty} |R_n| = \lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1}$$

$$e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

So $\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad \text{on } (-\infty, \infty)}$

$$e^x = \left(e^{x/2}\right)^2 = \left(e^{x/2^n}\right)^{2^n}$$