Math 243

~~Q5~~ $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=S$ $\quad S=\lim_{n\to\infty}(S_n)$

partial sum $\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\dots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$

$S_1=\left(\frac{1}{1}-\frac{1}{2}\right)$

$S_2=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)=1-\frac{1}{3}$

$S_3=\left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\frac{1}{4}\right)=1-\frac{1}{4}$

$S_4=\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)=1-\frac{1}{5}$

...

$S_n=\left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\cancel{\dots}+\left(\cancel{\frac{1}{n}}-\frac{1}{n+1}\right)$

$S_n=1-\frac{1}{n+1}$

So $S=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=\boxed{1}$

$\boxed{\sum_{n=0}^{\infty}ar^n}=a+ar+ar^2+\dots=\frac{a}{1-r}$, $|r|<1$

geometric series

ex $\boxed{\sum_{n=1}^{\infty}\left(\frac{e}{\pi}\right)^n}$?

$a\boxed{r^0}+a\boxed{r^1}+ar^2+ar^3+\dots$

$\left(\frac{e}{\pi}\right)^1+\left(\frac{e}{\pi}\right)^2+\left(\frac{e}{\pi}\right)^3+\left(\frac{e}{\pi}\right)^4+\dots$

$=\left(\frac{e}{\pi}\right)\left(\frac{e}{\pi}\right)^0+\frac{e}{\pi}\left(\frac{e}{\pi}\right)^1+\frac{e}{\pi}\left(\frac{e}{\pi}\right)^2+\frac{e}{\pi}\left(\frac{e}{\pi}\right)^3+\dots$

$a=e/\pi$

$r=e/\pi<1$

$\sum_{n=1}^{\infty}\left(\frac{e}{\pi}\right)^n=\boxed{\frac{e/\pi}{1-e/\pi}}$ b/c $e/\pi<1$

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots$$

or

Ex) $$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

b)

$$\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n = \left[\sum_{n=0}^{\infty} \left(\frac{e}{\pi}\right)^n\right] - 1 = \frac{1}{1 - e/\pi} - 1$$

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ex) $$\sum_{n=5}^{\infty} 3\left(\tfrac{1}{2}\right)^n$$

another idea

$$= \left[\sum_{n=0}^{\infty} 3\left(\tfrac{1}{2}\right)^n\right] - \left[3 + 3\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right)^2 + 3\left(\tfrac{1}{2}\right)^3 + 3\left(\tfrac{1}{2}\right)^4\right]$$

$$= \sum_{n=0}^{\infty} 3\left(\tfrac{1}{2}\right)^n - \sum_{n=0}^{4} 3\left(\tfrac{1}{2}\right)^n$$

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ex) $$\sum_{n=1}^{\infty} \frac{x^{2n}}{3^n} = \sum_{n=1}^{\infty} \left(\frac{x^2}{3}\right)^n$$ geometric series

$$\sum_{n=1}^{\infty} a r^{n-1}$$

$$\sum_{n=1}^{\infty} \underbrace{\left(\frac{x^2}{3}\right)}_{=a}\underbrace{\left(\frac{x^2}{3}\right)^{n-1}}_{=r} = \frac{a}{1-r} = \frac{(x^2/3)}{1-(x^2/3)} \quad \text{if } \left|\frac{x^2}{3}\right| < 1$$

$$= \frac{x^2}{3-x^2} \quad \text{if } x^2 < 3$$

$$\text{if } -\sqrt{3} < x < \sqrt{3}$$

$$\frac{1}{3}x^2 + \frac{1}{9}x^4 + \frac{1}{27}x^6 + \frac{1}{81}x^8 + \cdots = \frac{x^2}{3-x^2} \quad \text{if } -\sqrt{3} < x < \sqrt{3}$$

ex

$$\sum_{n=0}^{\infty} \frac{(-4)^{3n}}{5^{n-1}}$$ geometric?

$$\boxed{\sum_{n=1}^{\infty} a r^{n-1}} \quad |r|<1$$

$$= \sum_{n=0}^{\infty} \frac{\left[(-4)^3\right]^n}{\frac{5^n}{5}}$$

$$\boxed{\sum_{n=0}^{\infty} a r^n} = \frac{a}{1-r}$$

$a + ar + ar^2 + \ldots$

$$= \sum_{n=0}^{\infty} 5 \left(\frac{-64}{5}\right)^n = \sum_{n=0}^{\infty} (-1)^n\, 5 \left(\frac{64}{5}\right)^n$$

$|r| \not< 1 \Rightarrow$ divergent

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ex

$$\sum_{n=2}^{\infty} \frac{(-3)^n}{2^{2n+1}} = \sum_{n=0}^{\infty} \frac{(-3)^{n+2}}{2^{2(n+2)+1}} = \sum_{n=0}^{\infty} \frac{(-3)^{n+2}}{2^{2n+5}}$$

$$= \sum_{n=0}^{\infty} \frac{(-3)^2 (-3)^n}{(2)^5 (\cancel{2})^{2n}} = \sum_{n=0}^{\infty} \frac{9\,(-3)^n}{32\,(4)^n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{9}{32}\right)\left(-\frac{3}{4}\right)^n = \boxed{\frac{9/32}{1+3/4}}$$

b/c $\sum_{n=0}^{\infty} a r^n = \frac{a}{1-r}$ $\quad |r|<1$

11.8 Power Series

$$C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots = \sum_{n=0}^{\infty} C_n x^n$$

ex $C_i = 1$ $\quad 1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} \underset{a}{(1)}\, \underset{r}{x}^n = \frac{1}{1-x}$ if $|x|<1$

So: $1 + x + x^2 + x^3 + \dots = \dfrac{1}{1-x}$ only when $|x| < 1$

Power Series $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$

converges for some x's

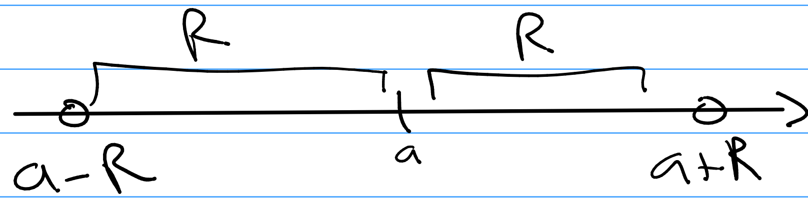
diverges for others

→ domain of f is only x's that power series converges.

Power Series in General (power series about $x=a$)

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

Convergence?



(1) R : radius of convergence

Points $x = a-R$, $x = a+R$ are special cases to figure out conv / div.

(2) **Interval of convergence**

a) conv $[a-R, a+R]$

b) conv $(a-R, a+R)$

c) conv $[a-R, a+R)$

d) conv $(a-R, a+R]$

$\boxed{\text{th}^n}$ $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are 3 possibilites

1. only converges @ $x=a$ ($R=0$)
2. converges for all x ($R=\infty$)
3. converges for x's between $a-R$, $a+R$
   (R is a finite pos number)
   (end points must be studied)

$\boxed{\text{ex}}$ $\sum_{n=0}^{\infty} \frac{1}{n!}x^n$

use ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$
a) $L<1$, abs conv
b) $L>1$, div.
c) $L=1$??

$$\to \lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)!}x^{n+1}}{\frac{1}{n!}x^n}\right| = \lim_{n\to\infty}\frac{\cancel{n!}}{(n+1)\cancel{!}}\frac{|x|^{n+1}}{\cancel{|x|^n}} = \lim_{n\to\infty}\frac{1}{n+1}|x| = 0$$

So $L=0<1 \to$ so $\sum_{n=0}^{\infty}\frac{1}{n!}x^n$ is abs. conv.
for $R=\infty$

$\boxed{\text{so}}$ $f(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots$

domain (abs. conv.) is all x's

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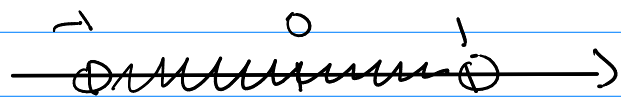
ex) $\sum_{n=1}^{\infty} \frac{1}{2n-1} x^n = x + \frac{1}{3}x^2 + \frac{1}{5}x^3 + \frac{1}{7}x^4 + \dots$

use ratio test. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{1}{2(n+1)-1}\, x^{n+1}}{\frac{1}{2n-1}\, x^n} \right|$

$= \lim_{n\to\infty} \frac{2n-1}{2n+1} \frac{|x|^{n+1}}{\cancel{|x|^n}} = \lim_{n\to\infty} \left(\frac{2n-1}{2n+1}\right)|x| = 1\cdot|x| = L$

$L = |x| < 1$ abs. conv. $\rightarrow -1 < x < 1$



$R = 1$

check: $x = 1$ and $x = -1$

$\sum_{n=1}^{\infty} \frac{1}{2n-1} x^n$:

$\sum_{n=1}^{\infty} \frac{1}{2n-1} (1)^n = \boxed{\sum_{n=1}^{\infty} \frac{1}{2n-1}}$

like a div $\frac{1}{n}$

(show divergent)

$\sum_{n=1}^{\infty} \frac{1}{2n-1} (-1)^n = \boxed{\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{2n-1}\right)}$

like a conv. alt. series

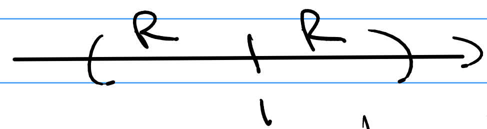
(show convergent)

$\therefore$ Interval of convergence

all $x \in [-1, 1)$

ex

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n}(x-1)^n = \frac{-1}{1\cdot 2}(x-1) + \frac{1}{3\cdot 4}(x-1)^2 - \frac{1}{5\cdot 8}(x-1)^3 + \dots$$



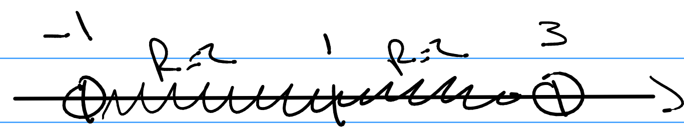
use ratio test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{(2n+1)2^{n+1}}(x-1)^{n+1}}{\frac{1}{(2n-1)2^n}(x-1)^n}\right|$

$$= \lim_{n\to\infty} \frac{2n-1}{2n+1}\,\frac{\cancel{2^n}}{2^{n+1}}\,\frac{|x-1|^{n+1}}{\cancel{|x-1|^n}}$$

$$= \lim_{n\to\infty}\left(\frac{2n-1}{2n+1}\right)\cdot\frac{1}{2}\cdot|x-1| = \lim_{n\to\infty}\left(\frac{2-\cancel{1/n}^{\,0}}{2+\cancel{1/n}^{\,0}}\right)\cdot\frac{1}{2}|x-1|$$

So $L = \boxed{\frac{|x-1|}{2} < 1}$ for abs. conv.

$\to |x-1| < 2 = R$



$\boxed{R=2}$

Interval?

$x=-1$ and $x=3$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n}(x-1)^n$:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)\cancel{2^n}}(-\cancel{2})^n$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{(2n-1)}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{(2n-1)} = \sum_{n=1}^{\infty} \frac{1}{2n-1}$$

show div. by comparison to $\frac{1}{n}$

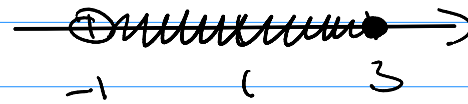
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)\cancel{2^n}}(\cancel{2})^n$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{1}{2n-1}$$

Show this is a conv. alt. series

$\boxed{\text{So}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)2^n}(x-1)^n$ $\quad R = 2$

Interval $(-1, 3]$



So we have found $f(x) = c_0 + c_1x + c_2x^2 + \dots$
and their domain

Intervals of convergence.

We had one special example:

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x} \quad \text{only on } |x| < 1$$

We can turn it around and see

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad \text{on } |x| < 1$$

$$\frac{1}{1-\boxed{x}} = 1 + \boxed{x} + \boxed{x}^2 + \boxed{x}^3 + \dots \quad \text{on } |\boxed{x}| < 1$$

$\boxed{\text{ex}}$ $$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - x^5, \dots$$

on $|x| < 1$

ex) $\dfrac{x^3}{2-x^2}$ Know: $\boxed{\dfrac{1}{1-\square}} = 1+\square+\square^2+\square^3+\dots$ on $|\square|<1$

$$\frac{x^3}{2-x^2} = \frac{(x^3/2)}{1-(x^2/2)} = \frac{x^3}{2}\left[\boxed{\frac{1}{1-[x^2/2]}}\right]$$

$$= \frac{x^3}{2}\left(1+\left(\frac{x^2}{2}\right)+\left(\frac{x^2}{2}\right)^2+\left(\frac{x^2}{2}\right)^3+\dots\right) \text{ on } \left|\frac{x^2}{2}\right|<1$$

$$= \frac{x^3}{2}+\frac{x^5}{2^2}+\frac{x^7}{2^3}+\frac{x^9}{2^4}+\dots \text{ on } |x^2|<2$$

$$\frac{x^3}{2-x^2} = \sum_{n=1}^{\infty} \frac{1}{2^n} x^{2n+1} \text{ on } -\sqrt{2}<x<\sqrt{2}$$

So use $\dfrac{1}{1-\square} = 1+\square+\square^2+\dots$ on $|\square|<1$ and <u>algebra</u> to make new equalities (see above)

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or <u>use calculus</u>

Thm: if $f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$ has a radius on convergence $R>0$, f is differentiable on $(a-R, a+R)$ interval

1. $f'(x) = c_1+2c_2(x-a)+3c_3(x-a)^2+\dots$
$= \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$

② $\int f(x)dx = C + c_0(x-a) + \frac{c_1}{2}(x-a)^2 + \frac{c_2}{3}(x-a)^3 + \dots$

$$= C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$$

and the radius of convergence stays the same.

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ex $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ on $|x|<1$

Derivative $\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{d}{dx}\left\{(1-x)^{-1}\right\} = -1(1-x)^{-2}(-1)$

$$= \frac{1}{(1-x)^2}$$

$\frac{d}{dx}\left[1 + x + x^2 + x^3 + \dots\right] = 1 + 2x + 3x^2 + \dots$

So $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$ on $|x|<1$

$$= \sum_{n=0}^{\infty}(n+1)x^n \text{ on } |x|<1$$

Know: $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots$ on $|x|<1$

Integration $\int \frac{1}{1+x}dx = \ln|1+x|$

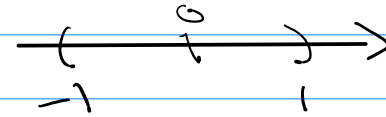
$\int (1 - x + x^2 - x^3 + \dots)dx = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$

$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$ on $|x|<1$

$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}x^n$ on $|x|<1$

So $\boxed{\ln(1+x)} = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$

on $-1 < x < 1$



$\ln(y) \quad 0 < y < 2$

$\overset{\parallel}{1+x}$

$y = 1+x \rightarrow x = y-1$

$$\boxed{\ln(y) = (y-1) - \frac{1}{2}(y-1)^2 + \frac{1}{3}(y-1)^3 - \frac{1}{4}(y-1)^4 + \cdots \quad \text{on } 0 < y < 2}$$

$$\ln(1.7) = .7 - \frac{1}{2}(.7)^2 + \frac{1}{3}(.7)^3 - \frac{1}{4}(.7)^4 + \cdots$$

$x = 1.7 - 1 = .7$

Next time: