Math 243

$\boxed{\text{Ch 11}}$ seq $\{a_n\}_{n=1}^{\infty} \to a_1, a_2, a_3, \dots$

$\lim_{n\to\infty} a_n = ?$ ① Know $\boxed{\text{ex}}$ $\lim_{n\to\infty} \frac{1}{n} = 0$

or

② $f(x)$ such that $f(n) = a_n$

$\lim_{x\to\infty} f(x) = L = \lim_{n\to\infty} a_n$

Series: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$

tests for divergence / convergence

① Divergence Test

$\lim_{n\to\infty} a_n \neq 0$ or dne

$\Rightarrow \sum a_n$ diverges

$\boxed{\text{Convergence of}}$ $\sum a_n$ means $a_1 + a_2 + a_3 + \dots = \boxed{S}$

$\uparrow$ finite number.

How? let $S_n = a_1 + a_2 + \dots + a_n$

$\lim_{n\to\infty} S = S$

$\uparrow$ seq. of partial sums

or we can approximate $S \approx S_n$ for a "large" n given some $\sum a_n$, test if it does converge (and when)

$\boxed{\text{Tests}}$ ① Integral Test if $f(n)=a_n$

and $f(x)$ is cont, pos, and decreasing on $[1,\infty)$

then $\int_1^\infty f(x)dx$ behaves like $\sum_{n=1}^{\infty} a_n$

② p-series test $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p \le 1$

converges if $p > 1$

③ Comparison Test, a_n, b_n are positive

$a_n \le b_n$ for all n

① if $\sum a_n$ diverges $\to \sum b_n$ diverges

② if $\sum b_n$ converges $\to \sum a_n$ converges

④ limit comparison a_n, b_n are positive

if $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ (positive finite number)

then the behave alike.

⑤ Alternating series

b_n is pos

$$\sum (-1)^n \boxed{b_n}$$

if $\lim_{n\to\infty} b_n \to 0$ → alt. series converges.

Ex5 $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}} = \frac{1}{2\sqrt{2^2-1}} + \frac{1}{3\sqrt{3^2-1}} + \frac{1}{4\sqrt{4^2-1}} + \dots$

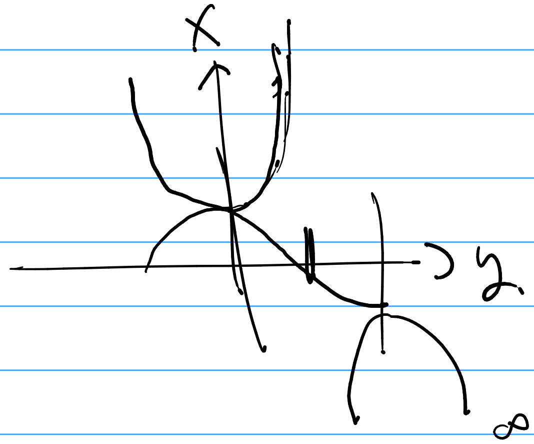
① <u>divergence test</u> $\displaystyle\lim_{n\to\infty} \frac{1}{n\sqrt{n^2-1}} = 0$ (test fails)

$\displaystyle\lim_{x\to\infty} \frac{1}{x\sqrt{x^2-1}} = 0$ ↳ <u>don't know</u>

② <u>integral test.</u> $\displaystyle\int_2^{\infty} \frac{1}{x\sqrt{x^2-1}}\,dx$

$$\lim_{b\to\infty} \int_2^b \frac{1}{x\sqrt{x^2-1}} \to \lim_{b\to\infty} \underline{\operatorname{arcsec}(b)} - \underline{\operatorname{arcsec}(2)} = \boxed{\text{Const.}}$$

<u>Converges</u>



So $\displaystyle\sum \frac{1}{n\sqrt{n^2-1}}$ $\boxed{\text{Conv.}}$

③ <u>Comparison test.</u> $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

$$\frac{1}{n\sqrt{n^2-1}} \underset{\underline{n \text{ is big.}}}{\sim} \frac{1}{n\sqrt{n^2}} = \boxed{\frac{1}{n^2}} \ \underline{\text{is conv.}}$$

guess convergent $\displaystyle\frac{1}{n\sqrt{n^2-1}} \le \frac{1 \leftarrow \text{make bigger}}{n\sqrt{n^2-1} \leftarrow \text{make smaller}}$?

④ <u>limit comparison test.</u> $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

<u>Guess</u> $\displaystyle\frac{1}{n\sqrt{n^2-1}} \overset{\text{big}}{\underset{n}{\sim}} \frac{1}{n\sqrt{n^2}} = \boxed{\frac{1}{n^2}} \leftarrow \text{conv.}$

find $\lim_{n\to\infty} \dfrac{\frac{1}{n\sqrt{n^2-1}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \dfrac{n^2}{n\sqrt{n^2-1}}$

$= \lim_{n\to\infty} \dfrac{n}{\sqrt{n^2-1}} = \lim_{n\to\infty} \dfrac{1}{\sqrt{1-\cancel{\frac{1}{n^2}}^{\to 0}}} = 1$ ← pos and finite

So $\frac{1}{n^2}$ behaves like $\frac{1}{n\sqrt{n^2-1}} \to \boxed{\text{conv}}$

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back to

Comparison test. $\sum_{n=1}^{\infty} \dfrac{1}{n\sqrt{n^2+3}}$

guess for large n $\dfrac{1}{n\sqrt{n^2+3}} \sim \dfrac{1}{n\sqrt{n^2}} = \dfrac{1}{n\cdot|n|} = \overset{\text{conv.}}{\boxed{\dfrac{1}{n^2}}}$

if you guess convergent $\dfrac{1}{n\sqrt{n^2+3}} \le ? \le ? \le \boxed{}$

↑ know this is conv.

ex $\dfrac{1}{n\sqrt{n^2+3}} \le \dfrac{1}{n\sqrt{n^2+0}} = \dfrac{1}{n\sqrt{n^2}} = \boxed{\dfrac{1}{n^2}}$ is conv.

conv
$>0 \to \sum \frac{1}{n\sqrt{n^2+3}}$
also conv.

Alt. series test? $\sum_{n=2}^{\infty} \dfrac{1}{n\sqrt{n^2-1}}$ ← not alternating

So can't use this test

ex $\frac{1}{\ln(3)} - \frac{1}{\ln(4)} + \frac{1}{\ln(5)} - \frac{1}{\ln(6)} + \cdots$

$= \sum_{n=1} (-1)^{n-1} \underline{\frac{1}{\ln(n+2)}}$ is an alternating series.

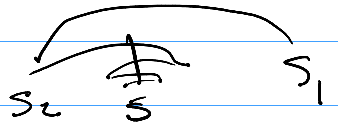
So $\lim_{n\to\infty} \frac{1}{\ln(n+2)} = 0$

$\lim_{x\to\infty} \frac{1}{\ln(x+2)} = \frac{1}{\infty} = 0$

and so $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\ln(n+2)}$ is convergent.

ex $S_6 = \frac{1}{\ln(3)} - \frac{1}{\ln(4)} + \frac{1}{\ln(5)} - \frac{1}{\ln(6)} + \frac{1}{\ln(7)} - \frac{1}{\ln(8)}$

= 0.285116009597483 → error < 0.455119613313419



S_6's error < $\frac{1}{\ln(9)}$

11.6 Absolute Convergence → convergence of the absolute value of the series.

Def when $\sum |a_n|$ converges

we say $\sum a_n$ is absolutely convergent.

ex $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by p-series test

but $\sum_{n=1}^{\infty} (-1)^{n+1} \boxed{\frac{1}{n}} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

by alt. series test $\lim_{n\to\infty} \frac{1}{n} = 0$

so alt. harmonic does converge.

Consider absolute convergence.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \xrightarrow[\text{value}]{\text{abs}} \sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{n} \right|$$

$$\parallel$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.}$$

→ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ is convergent
but not absolutely convergent

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ex) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$

① alt. series test $\lim_{n\to\infty} \frac{1}{n^2} = 0$

→ converges

② abs value $\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{1}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is
a convergent
p-series

→ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$ is absolutely
convergent

Thm if a series is absolutely convergent → it is convergent as well.

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abs. conv vs **conv**

not conv. and not abs. conv.	convergent and not abs. conv.	convergent and absolutely convergent
ex $\sum_{i=1}^{n} (-1)^n$	ex $\sum_{i=1}^{n} (-1)^{n+1} \frac{1}{n}$	ex $\sum_{i=1}^{n} (-1)^{n+1} \frac{1}{n^2}$
not convergent (divergent)	conditionally convergent	abs. convergent

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tests for abs. convergence

Ratio Test

① $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$

if $L < 1$ then $\sum a_n$ is absolutely conv.

② if $L > 1$ or $L = \infty$ then $\sum a_n$ is divergent

③ $L = 1$, then test fails → do something else.

Note: $a_n = f(n)$

$a_{n+1} = f(n+1)$

ex $a_n = 3n + n^2$

$a_{n+1} = 3(n+1) + (n+1)^2$

$a_{☺} = 3(☺) + (☺)^2$

ex $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$

ratio test $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{(n)^2}} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2}$

$= \lim_{n\to\infty} \frac{n^2}{n^2+2n+1} = \lim_{n\to\infty} \frac{1}{1+\cancelto{0}{2/n}+\cancelto{0}{1/n^2}} \boxed{=1}$

test fails

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root test $\lim_{n\to\infty} (|a_n|)^{1/n} = L$

① if $L < 1 \to \sum a_n$ is abs. conv.

② if $L > 1 \to \sum a_n$ is divergent

③ if $L = 1 \to$ test fails $\to$ do something else!

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ex $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{2^n n^3}$

ratio test $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{3^{n+1}}{2^{n+1}(n+1)^3}}{\frac{3^n}{2^n(n)^3}}$

$$= \lim_{n\to\infty} \frac{3^{n+1}}{2^{n+1}(n+1)^3} \cdot \frac{2^n (n)^3}{3^n} = \lim_{n\to\infty} \frac{3^{\cancel{n}+1}}{\cancel{3^n}} \frac{\cancel{2^n}}{2^{\cancel{n}+1}} \frac{n^3}{(n+1)^3}$$

$$= \lim_{n\to\infty} (3)\left(\frac{1}{2}\right)\left(\frac{1}{(1+1/n)^3}\right) = (3)\left(\frac{1}{2}\right)(1) = \frac{3}{2}$$

So $L = 3/2 > 1 \Rightarrow$ divergent

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ex $\displaystyle\sum_{n=1}^{\infty} \frac{n^{100}\,100^n}{n!}$

ratio test $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{(n+1)^{100}\,100^{n+1}}{(n+1)!}}{\frac{n^{100}\,100^n}{n!}}$

$$= \lim_{n\to\infty} \frac{(n+1)^{100}\,100^{n+1}}{(n+1)!} \cdot \frac{n!}{n^{100}\,100^n} = \lim_{n\to\infty} \frac{(n-1)^{100}}{n^{100}} \cdot \frac{100^{\cancel{n}+1}}{\cancel{100^n}} \cdot \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} \left(\frac{(1+1/n)^{100}}{1}\right)(100)\left(\frac{\cancel{n}(\cancel{n-1})(\cancel{n-2})\cdots(\cancel{1})}{(n+1)(\cancel{n})(\cancel{n-1})(\cancel{n-2})\cdots(\cancel{1})}\right)$$

$$= \lim_{n\to\infty} (1+1/n)^{100}(100)\left(\frac{1}{n+1}\right) = (1)(100)(0) = 0$$

$L = 0 < 1 \to \displaystyle\sum \frac{n^{100}\,100^n}{n!}$ is abs. conv.

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ex $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n} = \sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n^n}$

root test $\displaystyle\lim_{n\to\infty} (|a_n|)^{1/n} = \lim_{n\to\infty}\left(\frac{2^n}{n^n}\right)^{1/n} = \lim_{n\to\infty} \frac{2}{n} = 0$

so $L = 0 < 1 \rightarrow \sum \frac{(-1)^n}{n^2}$ is $\boxed{\text{abs. conv.}}$

$\boxed{\text{ex}}$ $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

$\boxed{\text{ratio test:}}$ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{1}{\ln(n+1)}}{\frac{1}{\ln(n)}} = \lim_{n\to\infty} \frac{\ln(n)}{\ln(n+1)} = 1$

→ consider $\lim_{x\to\infty} \frac{\ln(x)}{\ln(x+1)} = \lim_{x\to\infty} \frac{1/x}{1/(x+1)} = \lim_{x\to\infty} \frac{x+1}{x} = 1$

L=1 so test fails.

Alternating test.

↓

$\sum (-1)^n b_n$

$\lim_{n\to\infty} b_n = 0 \rightarrow \boxed{\text{conv.}}$

$\lim_{n\to\infty} \frac{1}{\ln(n)} = 0 \Rightarrow \sum (-1)^n \frac{1}{\ln(n)}$ is $\boxed{\text{conv.}}$

check the absolute value of the seq.

$$\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{\ln(n)}\right| = \sum_{n=2}^{\infty} \frac{1}{\ln(n)}$$

guess this diverges → compare to $\frac{1}{n}$

limit comparison test $\lim_{n\to\infty} \frac{\frac{1}{\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\ln n} = \infty$

b/c $\lim_{x\to\infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{1}{1/x} = \boxed{\infty}$

$\boxed{\text{test failed}}$

Comparison: guess divergent

$\frac{1}{\ln n} \geq \frac{1}{n} \to$ divergent

$\therefore \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ is conditionally convergent

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$\boxed{11.7}$ Strategy on $\sum a_n$

① Know div. and conv. series
geometric, p-series

② does $\lim_{n\to\infty} a_n \neq 0$ (divergence test)

③ alt. series. $\sum (-1)^n \boxed{b_n}$

$\uparrow$
check $\lim_{n\to\infty} b_n = 0$

④ $n!$ → use ratio test.

⑤ $(\ )^n$ in → maybe try root test

⑥ Integral test

⑦ limit comparison / comparison

ex $\sum_{n=1}^{\infty} 2n e^{-n^2+1}$

div. test $\lim_{n\to\infty} 2n e^{-n^2+1} = 0$

$$\lim_{x\to\infty} 2x e^{x^2+1} = \lim_{x\to\infty} \frac{2x}{e^{x^2-1}}$$

$$= \lim_{x\to\infty} \frac{2}{2x e^{x^2-1}} = 0$$

integral test $\int^{\infty} 2x e^{-x^2+1} \, dx$

$$= \lim_{b\to\infty} \int_1^b 2x e^{-x^2+1} \, dx = \lim_{b\to\infty} -\int_0^{-b^2+1} e^u \, du$$

let $u = -x^2+1$

$du = -2x\,dx$

$$= \lim_{b\to\infty} (-e^u)\Big|_0^{1-b^2} = \lim_{b\to\infty} -\cancel{e^{1-b^2}}^{\to 0} + 1 = 0+1 = 1$$

So $\int_1^{\infty} 2x e^{-x^2+1} \, dx$ is convergent

$\rightarrow$ $\sum_{n=1}^{\infty} 2n e^{-n^2+1}$ is also convergent

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ex $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n^2}\right)$

$$\lim_{n\to\infty} (-1)^n \cancel{\cos\left(\frac{1}{n^2}\right)}^{\to 1} = \lim_{n\to\infty} (-1)^n = \begin{matrix} -1 \\ \text{or} \\ 1 \end{matrix} \neq 0$$

d.n.e.

$\rightarrow$ $\sum (-1)^n \cos\left(\frac{1}{n^2}\right)$ is divergent.