Math 243

Divergence test of $\sum a_n$

if $\lim\limits_{n\to\infty} a_n \neq 0$ or dne $\rightarrow$ $\sum a_n$ is divergent

Properties of convergent series

given $\sum a_n$, $\sum b_n$ do converge, then so do

a) $\sum c\, a_n = c \sum a_n$

b) $\sum a_n + b_n = \sum a_n + \sum b_n$

c) $\sum a_n - b_n = \sum a_n - \sum b_n$

ex) $\sum\limits_{k=1}^{\infty} (\sin 100)^{k} = \sin(100) + (\sin(100))^2 + \cdots$

Know: $\boxed{\sum\limits_{k=1}^{\infty} a r^{k-1}} = a + ar + ar^2 + \cdots$

$= \dfrac{a}{1-r}$ when $|r| < 1$

$\sum\limits_{k=1}^{\infty} (\sin(100))(\sin(100))^{k-1}$ so $a = \sin(100)$

$r = \sin(100) \approx -.5$

$$= \boxed{\frac{\sin(100)}{1 - \sin(100)}}$$

Know geo series $\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$ when $|r|<1$

or $\sum_{k=0}^{\infty} ar^{k} = \frac{a}{1-r}$ when $|r|<1$

ex

$$S = \sum_{i=2}^{\infty}\left(\frac{1}{i+1} - \frac{1}{i-1}\right) \quad \text{vs} \quad \sum_{i=2}^{\infty}\left(\frac{2}{1-i^2}\right)$$

use partial fractions

Know (if converges) $S = \lim_{n\to\infty} S_n$ (partial sums)

$i=2$: $S_2 = \left(\frac{1}{3} - \frac{1}{1}\right)$

$i=2,3$: $S_3 = \left(\frac{1}{3} - \frac{1}{1}\right) + \left(\frac{1}{4} - \frac{1}{2}\right)$

$i=2,3,4$: $S_4 = \left(\frac{1}{3} - \frac{1}{1}\right) + \left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{5} - \frac{1}{3}\right)$

$i=2,3,4,5$: $S_5 = \left(\cancel{\frac{1}{3}} - \frac{1}{1}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{2}\right) + \left(\frac{1}{5} \cancel{- \frac{1}{3}}\right) + \left(\frac{1}{6} \cancel{- \frac{1}{4}}\right)$

⋮

$i=2,3,4,5,\dots,n$: $S_n = \left(\cancel{\frac{1}{3}} - \frac{1}{1}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{2}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{6}} - \cancel{\frac{1}{4}}\right) + \cdots$

$\cdots + \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n-3}}\right) + \left(\frac{1}{n} - \cancel{\frac{1}{n-2}}\right) + \left(\frac{1}{n+1} - \cancel{\frac{1}{n-1}}\right)$

$S_n = -\frac{1}{1} - \frac{1}{2} + \frac{1}{n} + \frac{1}{n+1}$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(-\frac{3}{2} + \cancelto{0}{\frac{1}{n}} + \cancelto{0}{\frac{1}{n+1}}\right) = \boxed{-3/2}$$

So $\boxed{\sum_{i=2}^{\infty} \frac{2}{1-i^2}} = \frac{2}{1-4} + \frac{2}{1-9} + \frac{2}{1-16} + \cdots$

$= \boxed{-1.5}$

Why? (1) $\sum a_n = S$ (know it converges)

→ for a large $n = N$ $\quad S_N \approx S$

error $= R_N = S - S_n$

(2) $f(x) = \sum_{k=0}^{\infty} a_k \boxed{x^k} = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$

$f(x) \approx \boxed{S_n}$ $\quad R_n = f(x) - S_n$

approx functions

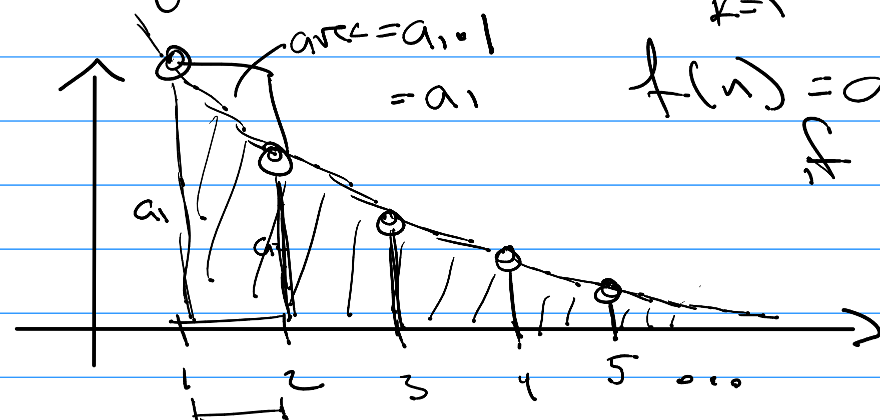
11.3 → 11.6 tests for $\sum a_n$ converging.

11.7 Strategies Section

For Convergence.

$\boxed{11.3}$ Integral Test for $\sum_{k=1}^{\infty} a_k$

Idea:



$f(n) = a_n$

if a_n are all pos.

f is cont., pos and dec

goal: $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \ldots$

$= a_1 \cdot 1 + a_2 \cdot 1 + a_3 \cdot 1 + \ldots$ = Sum of rectangles (area)

area under curve $\int_1^{\infty} f(x)\,dx$] behave alike.

area of rectangles $\sum_{k=1}^{\infty} a_k$

$\boxed{\text{Integral Test}}$ given $f(n) = a_n$, and $f(x)$ is cont, positive, decreasing function on $[1, \infty)$

then ① $\int_1^{\infty} f(x)\,dx$ converges $\Rightarrow \sum_{k=1}^{\infty} a_k$ converges

② $\int_1^{\infty} f(x)\,dx$ diverges $\Rightarrow \sum_{k=1}^{\infty} a_k$ diverges

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ex) $\sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$

$f(x) = \frac{1}{x}$ on $[1\ \infty)$ is it ① cont? yes

② pos? yes

③ dec? yes $f'(x) = \frac{-1}{x^2} < 0$

check $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{b \to \infty} \int_1^{b} \frac{1}{x}\,dx = \lim_{b \to \infty} \left(\ln|x| \Big|_1^b \right)$

$= \lim_{b \to \infty} \ln(b) = \infty$ diverges. $\therefore \sum_{k=1}^{\infty} \frac{1}{k}$ also diverges.

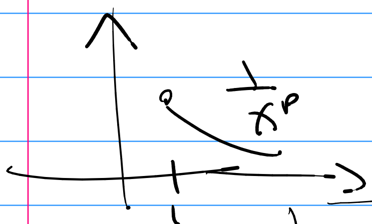
p-series test

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \qquad p>0$$

let $f(x) = \frac{1}{x^p}$ on $[1, \infty)$

1. $f(x)$ is cont.
2. $f(x)$ is positive
3. $f'(x) = -p\, x^{-p-1} = -\boxed{\frac{p}{x^{p+1}}}$

so $f'(x) < 0$ dec



so $\boxed{\int_1^{\infty} \frac{1}{x^p}\, dx}$ behaves like $\sum_{k=1}^{\infty} \frac{1}{k^p}$

$$\to \lim_{b\to\infty} \int_1^b x^{-p}\, dx = \lim_{b\to\infty} \left(\frac{1}{1-p} x^{-p+1}\right)\Big|_1^b \qquad (p \neq 1)$$

$$= \lim_{b\to\infty} \frac{1}{1-p}\left[b^{1-p} - 1\right]$$

important part is $\lim_{b\to\infty} b^{1-p}$

$\to p>1 \quad \frac{1}{b} \to 0$

$\to p<1 \quad b \to \infty$

previous $p=1$ is div.

so p-series test $\sum_{k=1}^{\infty} \frac{1}{k^p}$

convergent if $p>1$

divergent if $p \leq 1$

things we can sum or say convergent vs divergent.

① $\sum_{k=0}^{\infty} ar^k$ (geo series)

② $\sum_{k=0}^{\infty} a_k - a_{k+1}$ (variant of telescoping series)

③ $\sum_{k=1}^{\infty} \frac{1}{k^p}$ (p-series)

If you know $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \dots = S$ converges

So $\lim_{n \to \infty} S_n = S$ or each larger value of n, S_n gets closer to S.

We can approximate $\sum_{k=1}^{\infty} a_k \approx S_n$ for some n.

Error:

define $R_n = S - S_n$

Thm $\underbrace{\int_{n+1}^{\infty} f(x)\,dx}_{\text{lower bound}} \leq R_n \leq \underbrace{\int_{n}^{\infty} f(x)\,dx}_{\text{upper bound}}$

ex take 20 terms and find $S_{20} \approx S$

$$\int_{21}^{\infty} f(x)\,dx \leq R_{20} \leq \int_{20}^{\infty} f(x)\,dx$$

bounding S

$$S_n + \int_{n+1}^{\infty} f(x)\,dx \le S \le S_n + \int_{n}^{\infty} f(x)\,dx$$

ex's

given want error ≤ 0.0002

$$R_n \le 0.0002$$

$$R_n \le \boxed{\int_n^{\infty} f(x)\,dx \le 0.0002}$$

$$\sum_{i=1}^{\infty} \frac{-2}{i^2+2i} = -1.5 \quad \text{(from above)}$$

$S_n \approx S$ error ≤ 0.0001

$$\int_n^{\infty} \frac{-2}{x^2+2x}\,dx \le 0.0001$$

$$\lim_{b\to\infty} -2\int_n^{b} \frac{1}{x^2+2x}\,dx = ?$$

$$\frac{A}{x} + \frac{B}{x+2}$$

pause

$$\int \frac{1}{x^2+2x}\,dx = \int \frac{1}{x(x+2)}\,dx$$

can I do this? (yes) (see below)

check ① $\frac{1}{x(x+2)}$ cont on $[1,\infty)$? yes

② $\frac{1}{x(x+2)}$ pos on $[1,\infty)$? yes

③ $\frac{1}{x(x+2)}$ dec on $[1,\infty)$? Yes

$$f(x) = (x^2+2x)^{-1} \qquad f'(x) = -1\,(x^2+2x)^{-2}(2x+2)$$

$$= \underline{(-1)}\,\frac{(2x+2)}{(x^2+2x)^2} < 0$$

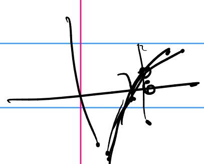
on $[1,\infty)$

So we can do $\displaystyle \lim_{b\to\infty} \boxed{2\int_n^b \frac{1}{x^2+2x}\,dx}$

$$= \lim_{b\to\infty} \ln(x+2) - \ln x \Big|_n^b = \lim_{b\to\infty} \cancel{\ln(b+2) - \ln(b)} \to 0$$

$$- \left[\ln(n+2) - \ln(n)\right]$$

$$= \ln n - \ln(n+2)$$



$$\ln\left(\frac{n}{n+2}\right) = 0.0001$$

$x = e^{0.0001}$

$$\frac{n}{n+2} = e^{0.0001}$$

$$n = (n+2)e^{0.0001}$$

$$n(1-e^{0.0001}) = 2e^{0.0001}$$

$$n = \left|\frac{2e^{0.0001}}{1-e^{0.0001}}\right|$$

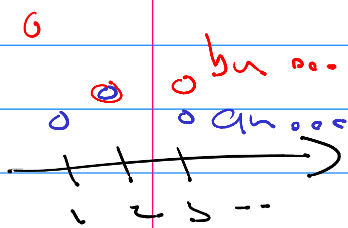
$$n \approx \underline{2\times 10^4}$$

(**Note**: Sign mistake @ start for this example :( )

More convergence tests.

$\boxed{11.4}$ Comparison tests.

Comparison test given Σa_n, Σb_n with a_n, b_n are positive and $a_n \leq b_n$ for all n



① if Σb_n converges $\to$ Σa_n converges

② if Σa_n diverges $\to$ Σb_n diverges

How to use this? on Σc_n

① guess Σc_n converges.

Find Σd_n that converges and $d_n \geq c_n$ for all n.

② guess Σc_n diverges

Find Σd_n the diverges and $d_n \leq c_n$ for all n.

$\boxed{\text{ex}}$ $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ acts like $\dfrac{1}{\sqrt{n}} = \dfrac{1}{n^{1/2}}$ divergent p-series.

guess this diverges $\dfrac{1}{\sqrt{n}-1} \geq \left(\begin{array}{c}\text{Something} \\ \text{that diverges}\end{array}\right)$

take $\frac{1}{\sqrt{n}-1} \geq \frac{1}{\sqrt{n}-1+1} = \frac{1}{\sqrt{n}}$

So $\sum \frac{1}{\sqrt{n}}$ diverges <u>and</u> $\frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{n}-1}$

so $\sum \frac{1}{\sqrt{n}-1}$ also diverges.

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ex $\sum_{k=1}^{\infty} \frac{1}{\sqrt{n}+1}$ guess divergent

find a smaller divergent series?

$$\frac{1}{\sqrt{n}+1} \geq \frac{1}{\sqrt{n}+\sqrt{n}} = \frac{1}{2\sqrt{n}}$$

show by integral test this is divergent

b/c $\sqrt{n} \geq 1$ when $n \geq 1$

$\therefore$ b/c $\sum \frac{1}{2\sqrt{n}}$ diverges and $\frac{1}{\sqrt{n}+1} \geq \frac{1}{2\sqrt{n}}$

$\therefore \sum \frac{1}{\sqrt{n}+1}$ is also divergent

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ex $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n} = \frac{e}{1} + \frac{\sqrt{e}}{2} + \frac{\sqrt[3]{e}}{3} + \frac{\sqrt[4]{e}}{4} + \cdots + \frac{e^{1/n}}{n} + \cdots$ all bigger than 1.

b/c $e^{1/n} > 1$

guess divergent $\frac{e^{1/n}}{n} \geq \frac{1}{n}$ which diverges.

so $\sum \frac{e^{1/n}}{n}$ diverges

$\boxed{\text{limit comparison test}}$ $\Sigma a_n, \Sigma b_n$ and a_n, b_n are positive

if $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c > 0$ and finite

then either both converge or both diverge.

$\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}+1}$ ~ guess this is like $\frac{1}{\sqrt{n}}$ which is divergent

$$\boxed{\text{test}} \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}+1}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{1}{1+1/\sqrt{n}^{\to 0}}$$

$$= \frac{1}{1+0} = 1$$ b/c it is a pos, finite number

→ they act alike

So $\boxed{\sum \frac{1}{\sqrt{n}+1} \text{ diverges}}$

$\boxed{\text{ex}}$ $\sum\limits_{n=1}^{\infty} \frac{n^2+n+1}{n^4+n}$ ~ large n ~ $\frac{n^2}{n^4} = \frac{1}{n^2}$ which is conv.

test $$\lim_{n\to\infty} \frac{\frac{n^2+n+1}{n^4+n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^4+n^3+n^2}{n^4+n} = \lim_{n\to\infty} \frac{1+1/n^{\to 0}+1/n^{2\to 0}}{1+1/n^{3\to 0}}$$

$= 1$ → pos. and finite

∴ alike.

$\boxed{\sum\limits_{n=1}^{\infty} \frac{n^2+n+1}{n^4+n} \text{ is convergent}}$

up to now all $\sum a_n$ ← must be **positive** (no subtraction allowed)

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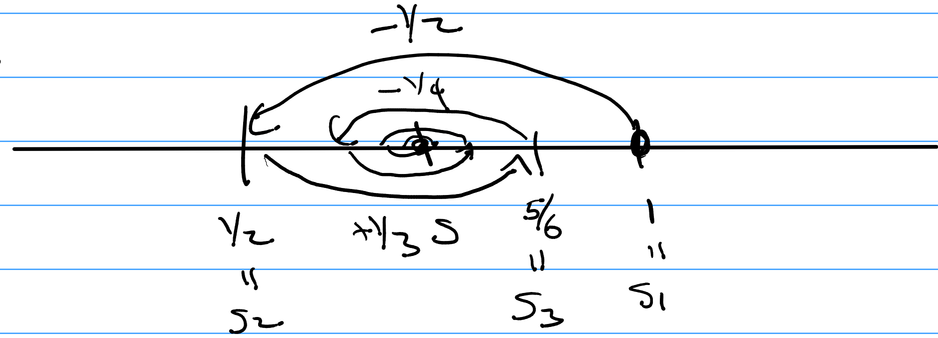
but what about? $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$

Subtraction/Addition in a series

1. **Alternating Series** $\sum_{k=\square}^{\infty} (-1)^k \boxed{b_k}$ ← always pos

$\boxed{\text{ex}}$ $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$

→ **convergent?**



$\boxed{\text{Alternating Series test}}$ $\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - \dots$

if ① $b_n \geq b_{n+1}$ (b_n are decreasing)

② $\lim_{n \to \infty} b_n = 0$

then $\sum (-1)^{n+1} b_n$ **converges.**

and error $= S_n - S \leq b_{n+1}$