Math 243

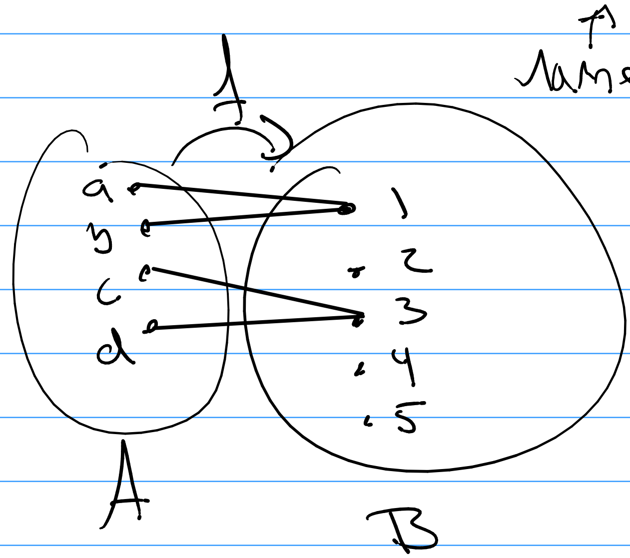
Ch 11 Sequences, Series, Power Series, Polynomial Approximations of functions.

11.1 Sequences (special <u>functions</u>)

$1, 4, 9, 16, 25, \dots$

Functional notation: $f: A \to B$

name — domain — codomain



ordered pairs

$(a,1)$

$(b,1)$

$(c,3)$

$(d,3)$

f — give a "rule"

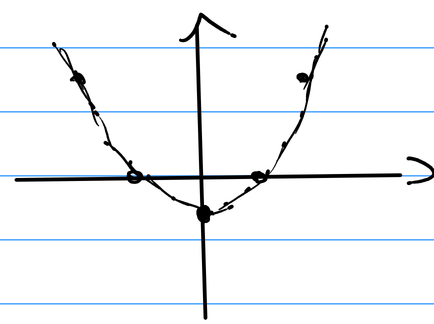
Calculus $\mathbb{R}$ is the set of all reals

$$f: \mathbb{R} \to \mathbb{R}$$

$$f(x) = x^2 - 1$$

x	y
-2	3
-1	0
0	-1
1	0

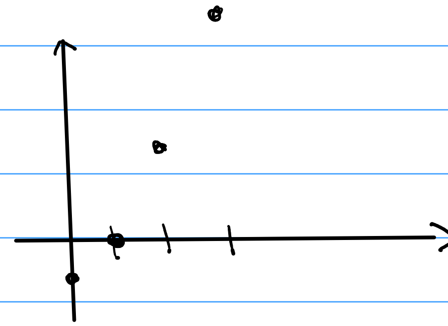
x	y
$\sqrt{2}$	1
$\sqrt{7}$	6



Seq f : (subset) Integers $\to$ Reals

function: $f(n) = n^2 - 1$ $\quad n = 0, 1, 2, 3, \dots$

n	n^2-1	
0	-1	$(0,-1)$
1	0	$(1,0)$
2	3	$(2,3)$
3	8	$(3,8)$
	4	



Seq: $-1, 0, 3, 8, \dots$

Seq. Notation: $f(n) = a_n$

$$\{a_n\}_{n=0}^{\infty} \to a_0, a_1, a_2, a_3, \dots$$

ex $\{n^2-1\}_{n=0}^{\infty} \to -1, 0, 3, 8, 15, \dots$

rule (function) that creates the seq

the seq.

$$\{n+2\}_{n=3}^{\infty} \to 5, 6, 7, 8, \dots$$

$$\{n^2\}_{n=1}^{\infty} \to 1, 4, 9, 16, 25, 36, \dots$$

$$\{n!\}_{n=0}^{\infty} \to 0!, 1!, 2!, 3!, 4!, \dots$$

$$1, 1, 2, 6, 24, 120,$$

if a formula for a seq is just a function..

ex $a_n = n!$ or $a_n = n^2 - 1$ or $a_n = 3n+4$

$\{3n+4\}_{n=0}^{\infty} \to \underset{a_0}{4}, \underset{a_1}{7}, \underset{a_2}{10}, \underset{a_3}{13}, \ldots, \underset{a_{1000}}{3 \cdot 1000 + 4} = \boxed{3004}$

$\boxed{\text{closed}}$ form for seq

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vs $\boxed{\text{open}}$ or recursive form.

$4, 7, 10, 13, 16, 19, \ldots$

$+3 \quad +3 \quad +3 \quad +3$

Basis: $a_0 = 4$

recursive rule (Inductive) $a_n = a_{n-1} + 3 \quad n = 1, 2, 3, \ldots$

ex $\underbrace{\boxed{a_1 = 2}}_{\text{basis}} \quad \underbrace{\boxed{a_n = 2a_{n-1} - 1 \quad n = 2, 3, 4, \ldots}}_{\text{recursive rule}}$

seq: $2, 3, 5, 9, 17, \ldots$

ex $\boxed{f_1 = 1, f_2 = 1} \quad \boxed{f_n = f_{n-1} + f_{n-2} \quad n = 3, 4, \ldots}$

seq: $1, 1, 2, 3, 5, 8, 13, 21, \ldots$

$(a+b)^3$

$a^3+3a^2b+3ab^2+b^3$

1	1
1 1	2
1 2 1	4
1 3 3 1	8
1 4 6 4 1	16

$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$

$\frac{4!}{3!\,1!}$ $\qquad$ $\frac{4!}{2!\,2!}$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

1, 1, 2, 3, 5

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Rule (open/closed) → seq

Ex: $C_0=1,\ C_1=1$ $\quad$ $C_n = C_0C_{n-1}+C_1C_{n-2}+C_2C_{n-3}+\cdots+C_{n-1}C_0$

$n=2,3,\ldots$

1, 1, 2, 5, 14, ...

Catalan seq

$C_2 = 1\cdot1+1\cdot1=2$

$C_3 = 1\cdot2+1\cdot1+2\cdot1=5$

$C_4 = 1\cdot5+1\cdot2+2\cdot1+5\cdot1=14$

B Seq → rule?

(recognize?)

$n = 0, 1, 2, \ldots$

$\{n\} = 0, 1, 2, 3, \ldots$

$\{n^2\} = 0, 1, 4, 9, \ldots$

$\{n^3\} = 0, 1, 8, 27, \ldots$

$\{n!\} = 1, 1, 2, 6, 24, \ldots$

$\{2^n\} = 1, 2, 4, 8, 16, \ldots$

$\{3^n\} = 1, 3, 9, 27, 81, \ldots$

$\{(-1)^n\} = 1, -1, 1, -1, 1, \ldots$

arithmetic seq $\{an + b\} = b, b+a, b+2a, b+3a, \ldots$

geometric seq $\{ar^n\} = a, ar, ar^2, ar^3, \ldots$

Ex $-3, 2, -4/3, 8/9, -16/27, \ldots$

$-3, +2, -4/3, +8/9, -16/27, \ldots$

$$(-1)^{-1}\frac{2^0}{3^{-1}},\ (-1)^0\frac{2^1}{3^0},\ (-1)^1\frac{2^2}{3^1},\ (-1)^2\frac{2^3}{3^2},\ (-1)^3\frac{2^4}{3^3}, \ldots$$

$n=0 \quad n=1 \quad n=2 \quad n=3 \quad n=4$

$$\boxed{a_n = (-1)^{n-1}\frac{2^n}{3^{n-1}} \qquad n = 0, 1, 2, \ldots}$$

Same $\Leftrightarrow$ $$\boxed{a_n = (-1)^n \frac{2^{n-1}}{3^{n-2}} \qquad n = 1, 2, 3, \ldots}$$

Sol:

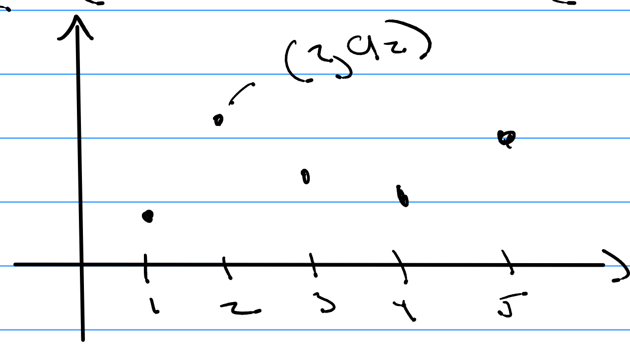
$0, 1, 3, 7, 15, \dots$ $\quad 2^n + ?$ $\quad -1$

$\quad 1 \quad 2 \quad 4 \quad 8$

$3, 4, 6, 10, 18, \dots$ $\quad 2^n + ?$ $\quad 2$

$\quad 1 \quad 2 \quad 4 \quad 8$

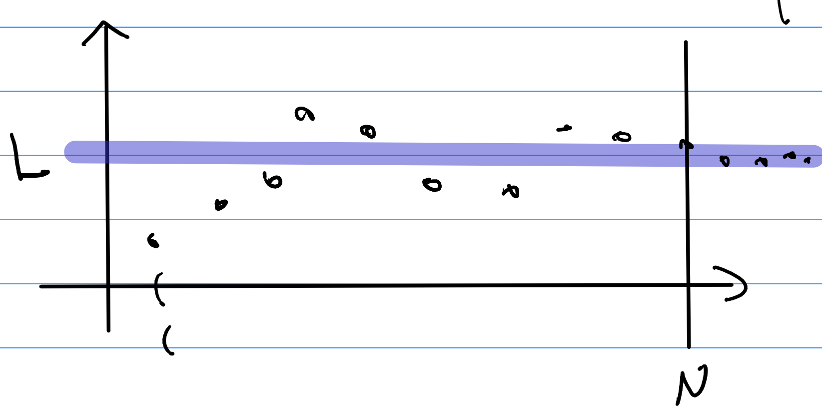
Note: (Visualize) $\quad \{a_n\}_{n=1}^{\infty}$



? limit ?

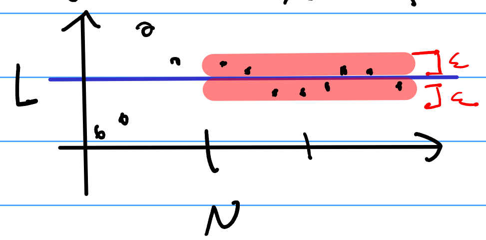
Def $\boxed{\lim_{n\to\infty} a_n = L}$ or $\boxed{a_n \to L \text{ as } n\to\infty}$

or $\boxed{a_n \xrightarrow[n\to\infty]{} L}$



Def $\lim_{n\to\infty} a_n = L$ if for all $\varepsilon > 0$ there exists an integer N such that

if $n > N$ then $|a_n - L| < \varepsilon$



if true $\lim_{n\to\infty} a_n = L$ (convergent)

otherwise divergent

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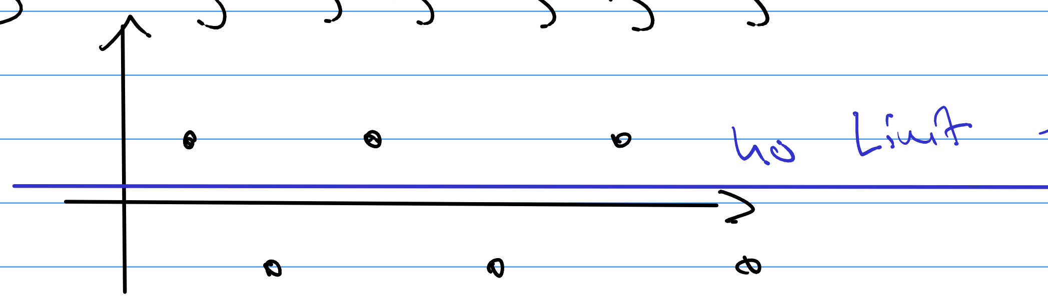
ex $1, -1, 1, -1, 1, -1, \dots$



no Limit → divergent

How to show convergence?

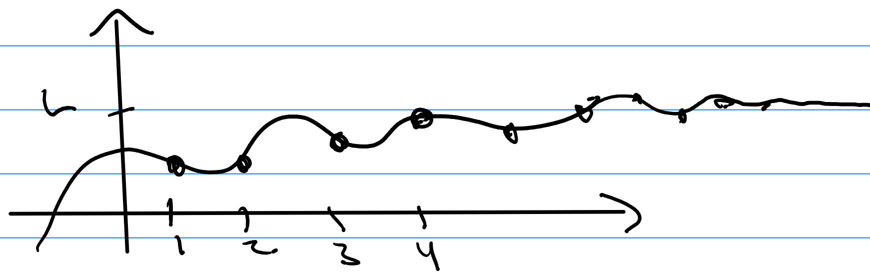
Note: we know how to deal with $f: \mathbb{R} \to \mathbb{R}$

ex $\lim_{x\to\infty} \frac{\sin x}{x} = 0$ $\quad -\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$, $\quad -\frac{1}{x} \to 0$, $\quad \frac{1}{x} \to 0$

$$\lim_{x\to\infty} \frac{x}{x^3+2x} = \lim_{x\to\infty} \frac{1}{3x^2+2} = 0$$

Thm if given $f(x): \mathbb{R} \to \mathbb{R}$

and $f(n) = a_n$



and $\lim\limits_{x\to\infty} f(x) = L$ then $\lim\limits_{n\to\infty} a_n = L$

Ex $a_n = \dfrac{3+n^2}{n+2n^2}$, $n=1,\dots$ $\qquad \lim\limits_{n\to\infty} \dfrac{3+n^2}{n+2n^2}$

Note: $f(x) = \dfrac{3+x^2}{x+2x^2}$ $\qquad f(n) = a_n$ ok

$$\lim_{x\to\infty} \frac{3+x^2}{x+2x^2} = \lim_{x\to\infty} \frac{\frac{3}{x^2}+1}{\frac{1}{x}+2} = \frac{1}{2}$$

So $\lim\limits_{n\to\infty} \dfrac{3+n^2}{n+2n^2} = \boxed{\dfrac{1}{2}}$

Limit Laws

① $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$

② $\lim\limits_{n\to\infty} c = c$

③ $\lim\limits_{n\to\infty} c\cdot a_n = c \lim\limits_{n\to\infty} a_n$

(4) $\lim_{n\to\infty} (a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$

(5) $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ $\quad \lim_{n\to\infty} b_n \neq 0$

(6) $\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p$ $\quad p > 1$, $a_n > 0$

(7) if f is a cont. function.

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$$

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ex $\lim_{n\to\infty} \sin(1/n) = \sin\left(\lim_{n\to\infty} \frac{1}{n}\right) = \sin(0) = 0$

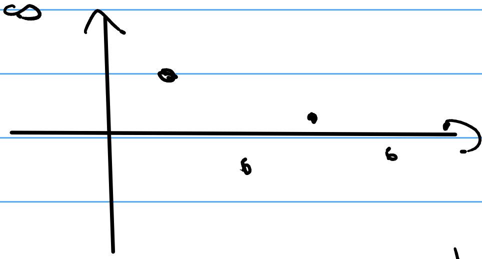
$\boxed{\lim_{n\to\infty} \frac{1}{n} = 0}$ why: $\lim_{x\to\infty} \frac{1}{x} = 0$

Squeeze thm $\quad a_n \le b_n \le c_n$ for $n \ge n_0$

if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \to \lim_{n\to\infty} b_n = L$

Corollary $\quad \lim_{n\to\infty} |a_n| = 0 \to \lim_{n\to\infty} a_n = 0$

ex $\lim_{n\to\infty} \frac{(-1)^n}{n}$ $\qquad \left\{\frac{(-1)^n}{n}\right\}_{n=1}^{\infty} \to -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \dots$



$f(x) = a_n$?

Consider $\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0$

$\Rightarrow \lim_{n\to\infty} \frac{(-1)^n}{n} = \boxed{0}$

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ex $\lim_{n\to\infty} \frac{3\sqrt{n}}{\sqrt{n}+2} = \boxed{3}$

Soln #1 $\quad \lim_{x\to\infty} \frac{3\sqrt{x}}{\sqrt{x}+2} = \lim_{x\to\infty} \frac{3}{1+2/\sqrt{x}} = 3$

Soln #2 $\quad$ Know: $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = \sqrt{\lim_{n\to\infty} \frac{1}{n}} = \sqrt{0} = 0$

$\lim_{n\to\infty} \frac{3\sqrt{n}}{\sqrt{n}+2} = \lim_{n\to\infty} \frac{3}{1+2\left(\frac{1}{\sqrt{n}}\right)} = \frac{3}{1+2\cdot 0} = \boxed{3}$

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ex $a_n = \sqrt[n]{2^{1+3n}} = \left(2^{1+3n}\right)^{1/n}$

$a_n = 2^{\frac{1}{n}+3}$

$\lim_{n\to\infty} 2^{3+\frac{1}{n}} = 2^{\lim_{n\to\infty}\left(3+\frac{1}{n}\right)} = 2^3 = \boxed{8}$

$\lim_{n\to\infty} a_n$ $\quad \{a_n\}_{n=1}^{\infty} \to 0, 1, 0, 0, 1, 0, 0, 0, 1, \ldots$

$\boxed{\text{diverges}}$

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ex) $\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r|<1 \\ 1 & \text{if } r=1 \\ \text{divergent} & |r|>1,\ r=-1 \end{cases}$

$\lim_{n\to\infty} r^n = \infty$

Special Sequences

A) Monotonic Sequences

1. $a_n < a_{n+1}$, $n \geq 1$ $\quad \{a_n\}$ is increasing

2. $a_n > a_{n+1}$, $n \geq 1$ $\quad \{a_n\}$ is decreasing

B) Boundedness

1. there exists M such that $a_n \leq M$ for all $n \geq 1$
   $\{a_n\}$ is bounded above

2. there exists m such that $m \leq a_n$ for all $n \geq 1$
   $\{a_n\}$ is bounded below.

3. bounded above and below $\to$ bounded

Th^m — Monotonic Seq Th^m

Every bounded, monotonic seq is convergent.

11.2 Series

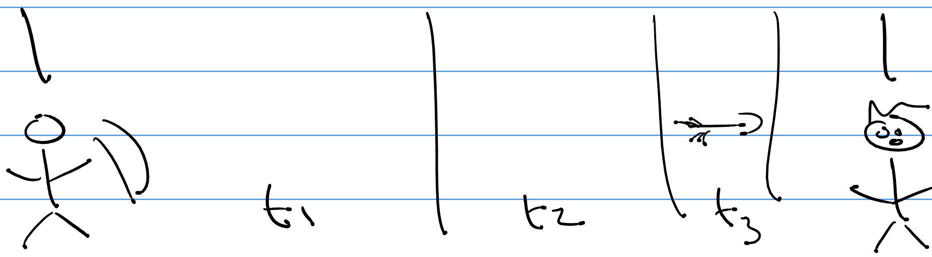
$p(x) = a_0 + a_1 x + a_2 x^2 + \dots$

↳ infinite sum of a seq.

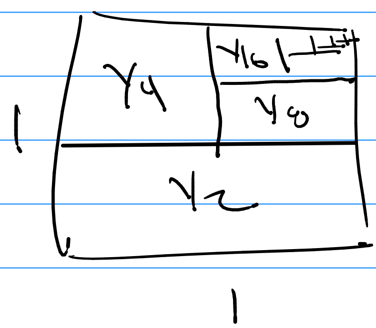
$\{a_n\}_{n=1}^{\infty} \to a_1, a_2, a_3, a_4, \dots$

Series: $a_1 + a_2 + a_3 + a_4 + \dots$

Notation: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$



$t = t_1 + t_2 + t_3 + \dots \underset{0}{\overset{?}{=}} \infty$



Area $= 1 = 1/2 + 1/4 + 1/8 + 1/16 + \dots$

$1 + 1 + 1 + 1 + \dots = \infty$

$\boxed{\text{ex}}$ $\sum_{i=1}^{\infty}(3i+1/i) = \underline{(3+1/1)} + \underline{(3\cdot 2+1/2)} + \underline{(3\cdot 3+1/3)} + \cdots$

How do add these?

(1) can't add ∞ things so...

(2) add some of them!

$\sum_{i=1}^{\infty} a_i$

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$\vdots$

$S_n = a_1 + a_2 + \cdots + a_n$

partial sums

Seq of partial sums $S_1, S_2, S_3, \ldots$

$\boxed{\text{Def}}$ if $\lim_{n\to\infty} S_n = S$ then $\sum_{i=1}^{n} a_i = S$

call $\sum a_i$ convergent

$\boxed{\text{ex}}$ $\sum_{i=0}^{\infty} 2(1/3)^i = 2 + 2(1/3) + 2(1/3)^2 + \ldots$

$\boxed{\text{geometric series}}$

$- \; S_n = 2 + \cancel{2(1/3)} + \cancel{2(1/3)^2} + \cancel{\cdots} + \cancel{2(1/3)^n}$

$\frac{1}{3} S_n = \cancel{2(1/3)} + \cancel{2(1/3)^2} + \cancel{2(1/3)^3} + \cdots \cancel{\cdots} + 2(1/3)^{n+1}$

$\frac{1}{3} S_n - S_n = 2(1/3)^{n+1} - 2$

$S_n(1/3 - 1) = 2\left[(1/3)^{n+1} - 1\right]$

$$S_n = 2\left[\frac{(1/3)^{n+1} - 1}{(1/3) - 1}\right]$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2\left[\frac{(1/3)^{n+1 \to 0} - 1}{(1/3) - 1}\right] = 2\left[\frac{-1}{-2/3}\right]$$

$$= \boxed{3}$$

Show $1 = 0.99999\ldots = 9(\tfrac{1}{10}) + 9(\tfrac{1}{10})^2 + 9(\tfrac{1}{10})^3 + \ldots$

$$- \quad S_n = 9(\tfrac{1}{10}) + 9(\tfrac{1}{10})^2 + \ldots + 9(\tfrac{1}{10})^n$$

$$(\tfrac{1}{10})S_n = 9(\tfrac{1}{10})^2 + 9(\tfrac{1}{10})^3 + \ldots + 9(\tfrac{1}{10})^{n+1}$$

$$\tfrac{1}{10}S_n - S_n = 9(\tfrac{1}{10})^{n+1} - 9(\tfrac{1}{10})$$

$$S_n = 9\left[\frac{(\frac{1}{10})^{n+1} - \frac{1}{10}}{\frac{1}{10} - 1}\right]$$

$$\lim_{n\to\infty} S_n = 9\left[\frac{0 - 1/10}{-9/10}\right] = \boxed{1}$$

Note: $\frac{1}{4} = 0.25 = 0.249999\ldots$

$\boxed{\text{geometric series}}$ $\displaystyle\sum_{i=0}^{\infty} ar^n = a + ar + ar^2 + \cdots$

$$\sum_{i=0}^{\infty} ar^n = \frac{a}{1-r} \quad \text{if} \quad |r| < 1$$

$$\text{diverges} \quad \text{if} \quad |r| \geq 1$$

Note $r = -1$ $\sum_{i=0}^{\infty} a(-1)^n = a - a + a - a + a - \ldots$

$S_1 = a$

$S_2 = 0$ $\{S_n\} \to a, 0, a, 0, a, 0, \ldots$

$S_3 = a$

$S_4 = 0$ $\boxed{\text{diverges}}$

$\vdots$

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ex Telescoping Series

$$\sum_{i=1}^{\infty} a_i - a_{i+1} = (a_1 - a_2) + (a_2 - a_3) + \ldots$$

When divergent? When convergent? later

if $\sum a_n$ converges $\to \lim_{n \to \infty} a_n = 0$

logic: $\square \to \triangle$ (contrapositive)

Same as not $\triangle \to$ not $\square$

Thm divergence test

if $\lim_{n\to\infty} a_n$ does not exist or $\neq 0$

$\to \sum a_n$ diverges

$\boxed{ex}$ $\sum_{i=1}^{n}(3n+4) = 7+10+13+\cdots$

$\lim_{n\to\infty} 3n+4 = \infty \neq 0 \to \boxed{\text{Div}}$

$\sum_{i=1}^{n}\left(\frac{n+2}{2n-3}\right) = \frac{3}{-1}+\frac{4}{1}+\frac{5}{3}+\cdots$

$\lim_{n\to\infty}\frac{n+2}{2n-3} = \frac{1}{2} \neq 0$ so $\boxed{\text{diverges}}$

$\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$

$\lim_{n\to\infty}\frac{1}{n} = 0$ (divergent test does not apply) ¿ don't know conv ?

$S_n = 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\cdots\frac{1}{16}\right)+\cdots+\frac{1}{n}$

$1+\frac{1}{2}+\underbrace{\left(\frac{1}{4}+\frac{1}{4}\right)}_{=1/2}+\underbrace{\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)}_{=1/2}+\left(1/2\right)+\cdots\ \infty$