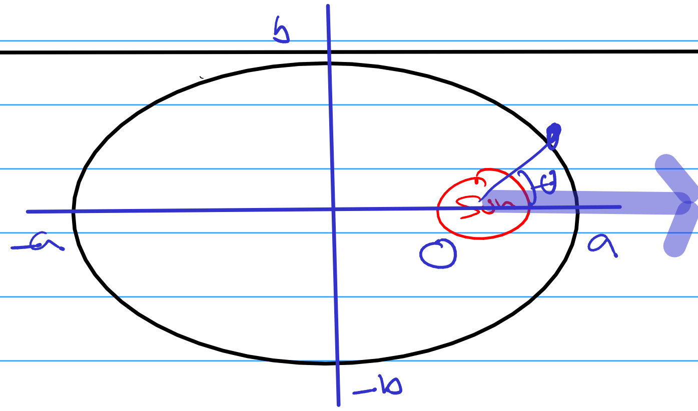
Math 243

Q's/ Solar system



$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

closest to sun $\theta = 0$ → $r_c = a(1-e)$

farthest from sun $\theta = \pi$ → $r_f = a(1+e)$

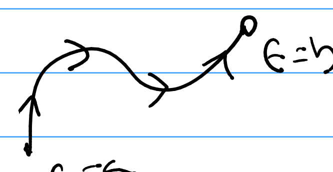
Exam 17 probs @ 10 pts each

160 pts = 100%

90 min's

10.1 Parametric Curves $(x(t), y(t))$ $t \in [a,b]$

(1 prob)



① Convert from $x(t), y(t)$ into cartesian eqn. (eliminate the parameter)

$x = t + 3$ $t = 0$ to $t = 4$

$y = t^2 - t + 1$

Make into function of x and plot.

→ $t = x - 3$ → $y = (x-3)^2 - (x-3) + 1$

$$y = x^2 - 6x + 9 - x + 3 + 1$$

$y' = 2x - 7$

$2x - 7 = 0$

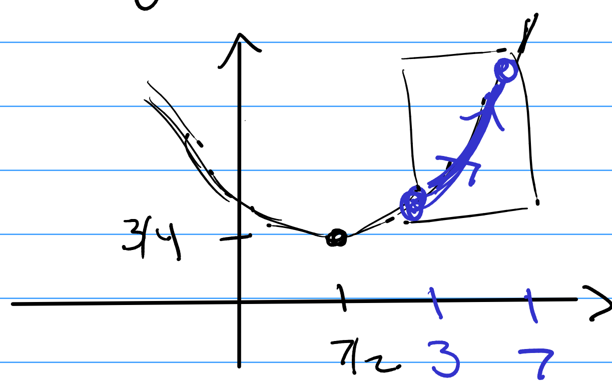
$x = 7/2$

$$y = x^2 - 7x + 13$$

$5^2/4 \qquad 49/4$

$$y = x^2 - 7x + (-7/2)^2 + \boxed{13} - \boxed{(7/2)^2}$$

$$y = (x - 7/2)^2 + 3/4$$



$x = t + 3 \qquad t \in [0, 4]$

$y = t^2 - t + 1$

$\boxed{10.2}$ Calculus of $x(t), y(t)$ (5 probs)

- dy/dx
- d^2y/dx^2
- areas
- arc length
- surface areas

① given $x(t), y(t)$

$\boxed{\text{find}}$ $\frac{dy}{dx}$, d^2y/dx^2

$\boxed{\text{ex}}$ $x(t) = 2t + 3$

$y(t) = t^2 - t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t - 1}{2} = \boxed{t - 1/2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}[t - 1/2]}{\frac{d}{dt}[2t + 3]} = \boxed{\frac{1}{2}}$$

Note:

$$\boxed{y'' = 1/2}$$

$$y' = \frac{1}{2}x + C$$

$$y = \frac{1}{4}x^2 + Cx + d$$

(2) $x(t), y(t)$

horz tangents?

vert. tangents?

slope @ t = something?

ex

$y(t) = \sin(2t)$ $\quad t \in [-\pi, \pi]$

$x(t) = t^2$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos(2t)}{2t} = \frac{\cos(2t)}{t}$$

horz. tangent $\rightarrow$ $\cos(2t) = 0$

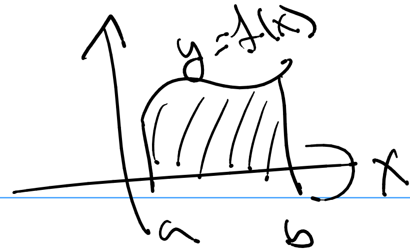
$$2t = \pi/2 \pm n\pi$$

$$t = \pi/4 \pm n\pi/2$$

$t = \boxed{-3\pi/4, -\pi/4, \pi/4, 3\pi/4}, \cancel{5\pi/4}, \cancel{7\pi/4},$

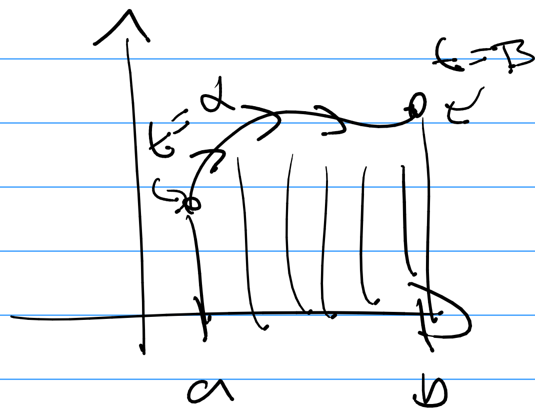
Vert tangent $\rightarrow$ $\boxed{t = 0}$

③ Area



$$A=\int_a^b y\,dx$$

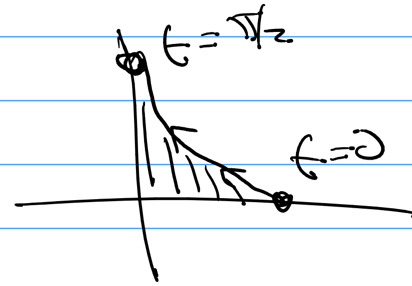
$x(t), y(t)$



$$A=\int_\alpha^\beta y(t)\,x'(t)\,dt$$

ex

$$x=\cos^2 t$$
$$y=\sin^3 t$$



$$A=\int_{\pi/2}^{0}\sin^3 t\; 2\cos t\,(-\sin t)\,dt$$

$$A=2\int_0^{\pi/2}\sin^4 t\,\cos t\,dt=2\int_0^1 u^4\,du$$

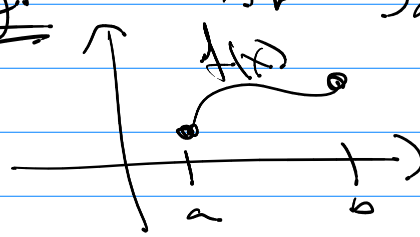
let $u=\sin t$

$du=\cos t\,dt$

$$A=\frac{2}{5}u^5\Big|_0^1=\boxed{\frac{2}{5}}$$

④ Arclength

$$AL=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$



$x(t), y(t)$ $\quad AL = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

ex $\quad x(t) = t\sin t \qquad t \in [0, 1]$

$y(t) = t\cos t$

$AL = \int_0^1 \sqrt{x'^2 + y'^2}\, dt \qquad x' = \sin t + t\cos t$

$y' = \cos t - t\sin t$

$$AL = \int_0^1 \left((\sin t + t\cos t)^2 + (\cos t - t\sin t)^2 \right)^{1/2} dt$$

$$AL = \int_0^1 \left(\boxed{\sin^2 t} + \boxed{2\cancel{t\sin t\cos t}} + \boxed{t^2\cos^2 t} + \boxed{\cos^2 t} - \boxed{2\cancel{t\sin t\cos t}} + \boxed{t^2\sin^2 t} \right)^{1/2} dt$$

$$AL = \int_0^1 (1+t^2)^{1/2}\, dt = \int_0^{\pi/4} \sqrt{\sec^2\theta}\, \sec^2\theta\, d\theta$$

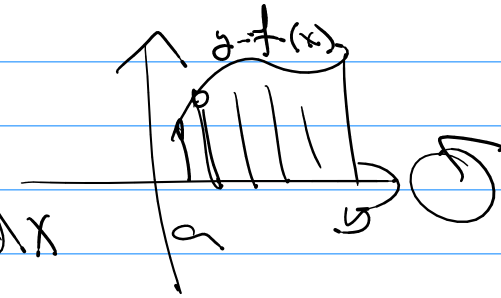
let $t = \tan\theta$ $\qquad = \int_0^{\pi/4} \sec^3\theta\, d\theta$

$dt = \sec^2\theta\, d\theta$

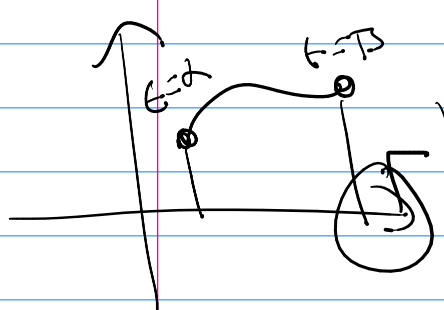
$= etc \ldots$

$1 + \tan^2\theta = \sec^2\theta$

(5) Surface of revolution



$$SA = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



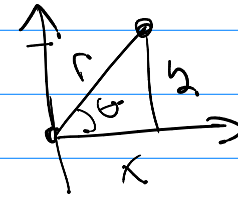
$y(t)$
$x(t)$

$$SA = \int_{\alpha}^{\beta} 2\pi y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

10.3 Polar Coordinates (3 probs)

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$
$$\tan\theta = y/x$$



① given $(x,y) \to (r,\theta)$

given $(r,\theta) \to (x,y)$

② dy/dx vert asym, horz. asym

ex given $r = 2\cos(2\theta)$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{d}{d\theta}[2\cos 2\theta \sin\theta]}{\frac{d}{d\theta}[2\cos 2\theta \cos\theta]}$$

$r\sin\theta$

$r\cos\theta$

$$\frac{dy}{dx} = \frac{-4\sin 2\theta \sin\theta + 2\cos 2\theta \cos\theta}{-4\sin 2\theta \cos\theta - 2\cos 2\theta \sin\theta}$$

horz tangents: $-4\sin 2\theta \sin\theta + 2\cos 2\theta \cos\theta = 0$

vert. tangents: $-4\sin 2\theta \cos\theta - 2\cos 2\theta \sin\theta = 0$

$\sin(2\theta) = 2\sin\theta\cos\theta$ (horz)

$\cos(2\theta) = \cos^2\theta - \sin^2\theta$ $\quad -8\sin^2\theta\cos\theta + 2\cos 2\theta\cos\theta = 0$

$= 1 - 2\sin^2\theta$ $\quad -2\cos\theta[\cancel{4\sin^2\theta} + 2 - \cancel{4\sin^2\theta})] = 0$

horz: $\cos\theta = 0$

QA

(3) given $r = f(\theta) \rightarrow$ find it's cartesian form.

$r = \cos\theta$ ~ x, y?

$\sqrt{x^2+y^2} = \frac{r\cos\theta}{r}$ $\qquad x = r\cos\theta$

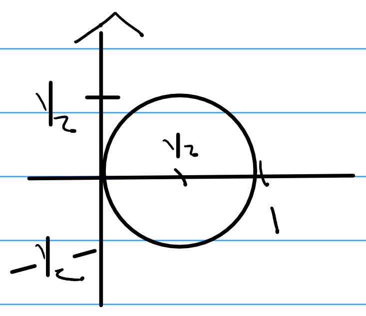
$\sqrt{x^2+y^2} = \frac{x}{\sqrt{x^2+y^2}} \rightarrow \boxed{x^2+y^2 = x}$

$x^2 - x + (-\tfrac{1}{2})^2 + y^2 = 0 + (-\tfrac{1}{2})^2$

$(x - \tfrac{1}{2})^2 + y^2 = \frac{1}{4}$

circle: center @ $(\tfrac{1}{2}, 0)$

$r = \tfrac{1}{2}$

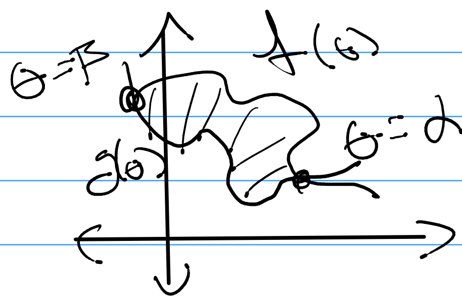


10.4 Areas, Arclength of $r = f(\theta)$ (3 probs)

(1) Area

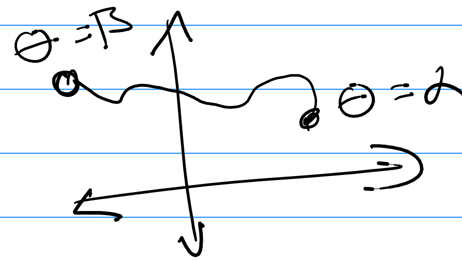
$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$

② Area between curves



$$A = \int_\alpha^\beta \tfrac{1}{2}f^2 - \tfrac{1}{2}g^2 \, d\theta$$

③ Arc length

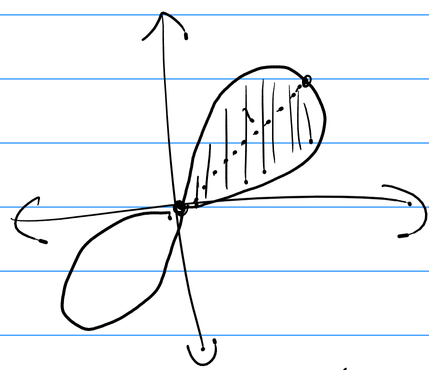


$x(\theta) = r\cos\theta$
$y(\theta) = r\sin\theta$
$AL = \int_\alpha^\beta \sqrt{x'^2 + y'^2} \, d\theta$

$$AL = \int_\alpha^\beta \sqrt{r^2 + (r')^2} \, d\theta$$

ex

$r^2 = \sin 2\theta$



$$A = \int_\alpha^\beta \tfrac{1}{2} r^2 \, d\theta = \tfrac{1}{2} 2 \int_0^{\pi/4} \sin 2\theta \, d\theta$$

$$A = \int_0^{\pi/4} \sin(2\theta)\,\cancel{d\theta} = \tfrac{1}{2}\int_0^{\pi/2} \sin u \, du$$

$u = 2\theta$
$du = 2d\theta$

$$= -\tfrac{1}{2}\cos u \Big|_0^{\pi/2}$$

$$= \cancel{\left(-\tfrac{1}{2}\cos \pi/2\right)} - \left(-\tfrac{1}{2}\cos 0\right)$$

$$= \boxed{1/2}$$

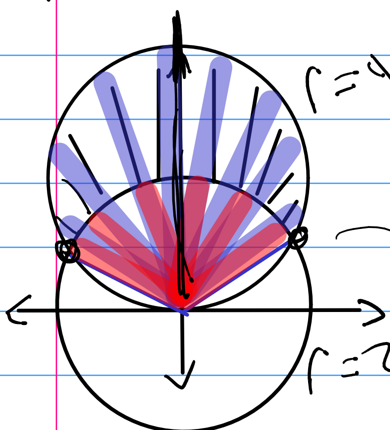
$$A = \int_0^{\pi/4} \sin 2\theta \, d\theta = \int_0^{\pi/4} 2\sin\theta\cos\theta \, d\theta$$

$u = \sin\theta$

$du = \cos\theta \, d\theta$

$$A = \int_0^{1/\sqrt{2}} 2u \, du = u^2 \Big|_0^{1/\sqrt{2}} = \boxed{\frac{1}{2}}$$

ex) inside $r_1 = 4\sin\theta$ outside $r_2 = 2$



$$A = \int_\alpha^\beta \left(\frac{1}{2}r_1^2 - \frac{1}{2}r_2^2\right) d\theta$$

$4\sin\theta = 2$

$\sin\theta = \frac{1}{2}$

$\theta = \pi/6$

$$A = 2\int_{\pi/6}^{\pi/2} \frac{1}{2}r_1^2 - \frac{1}{2}r_2^2 \, d\theta$$

$$A = \int_{\pi/6}^{\pi/2} (r_1^2 - r_2^2) \, d\theta = \int_{\pi/6}^{\pi/2} 16\sin^2\theta - 4 \, d\theta$$

$$= 4\int_{\pi/6}^{\pi/2} 4\sin^2\theta - 1 \, d\theta \qquad 1 - \cos 2\theta = 2\sin^2\theta$$

etc

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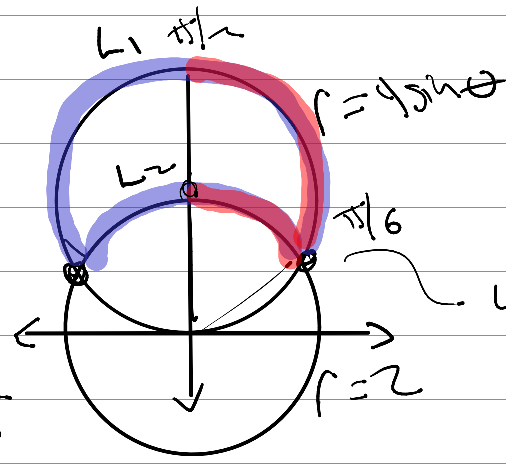
ex) Arc length of the perimeter of enclosed region

$$AL = \int_\alpha^\beta \sqrt{r^2 + (r')^2} \, d\theta$$



2 4π

$$L_1 = 2\int_{\pi/6}^{\pi/2} \left((4\sin\theta)^2 + (4\cos\theta)^2\right)^{1/2} d\theta$$

$$L_1 = 2\int_{\pi/6}^{\pi/2} \left(16\sin^2\theta + 16\cos^2\theta\right)^{1/2} d\theta$$

$$= 2\int_{\pi/6}^{\pi/2} 4\, d\theta = 8\int_{\pi/6}^{\pi/2} d\theta = \boxed{8\left(\pi/2 - \pi/6\right)}$$

$$= \boxed{\frac{8}{3}\pi}$$

$$L_2 = 2\int_{\pi/6}^{\pi/2} \left(2^2 + 0^2\right)^{1/2} d\theta$$

$$= 4\int_{\pi/6}^{\pi/2} = \boxed{\frac{4}{3}\pi}$$

$$AL = L_1 + L_2 = \frac{8}{3}\pi + \frac{4}{3}\pi = \boxed{4\pi}$$

$\boxed{10.5}$ Conic Sections (3 probs)

① mixed up x's y's $\longrightarrow$ $\boxed{Q?}$ is it $\rightarrow$ Parabola? ellipse? hyperbola?

$\boxed{\text{ex}}$ $x^2 - 2x + 2y^2 - 8y + 7 = 0$

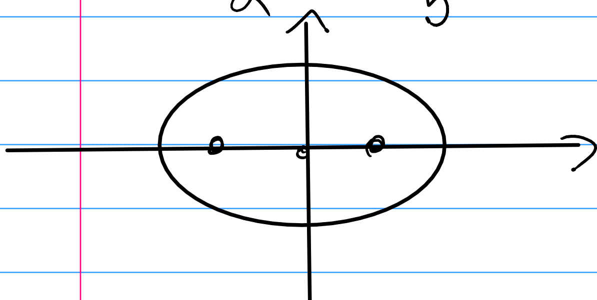
(2) given info about parabola, ellipse, or hyperbola

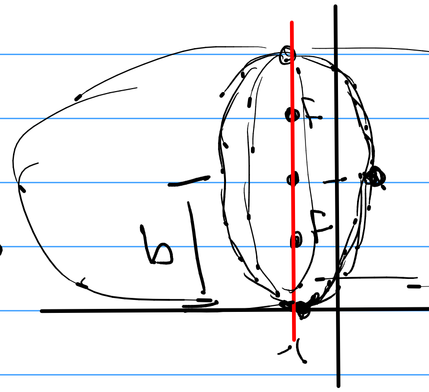
$\Rightarrow$ eqn

ex: ellipse; center $(-1,4)$; vertex $(-1,0)$

focus $(-1,6)$

standard: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$





$\frac{(x+1)^2}{a^2}+\frac{(y-4)^2}{4^2}=1$

one unknown: $\boxed{a}$

but our one point can't be used b/c

$\frac{\cancel{0}}{a^2}+\frac{(-4)^2}{4^2}=1$

$1=1$

eqn: $\boxed{\frac{(x+1)^2}{a^2}+\frac{(y-4)^2}{4^2}=1}$

for any a.

if we were given one more point on the curve.

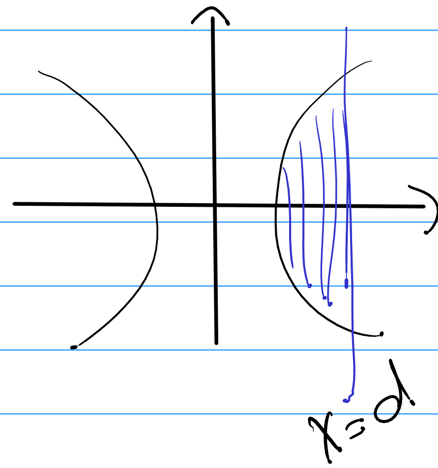
say $(1,4)$ is on the curve.

$$\frac{(1+1)^2}{a^2}+\cancel{\frac{(4-4)^2}{4^2}}=1 \rightarrow \frac{2^2}{a^2}=1 \rightarrow \boxed{a=2}$$

eqn $\boxed{\dfrac{(x+1)^2}{2^2} + \dfrac{(y-4)^2}{4^2} = 1}$

(3) Area inside a hyperbola and line.

ex



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

10.6 Conic Sections in Polar (2 probs)

$$r = \frac{ed}{1 \pm e\cos\theta} \qquad r = \frac{ed}{1 \pm e\sin\theta}$$

(1) given info → give eqn

(2) given eqn → give curve (plot)

$$r = \frac{3}{1 \boxed{+} 0.9\cos\theta}$$

e? d?