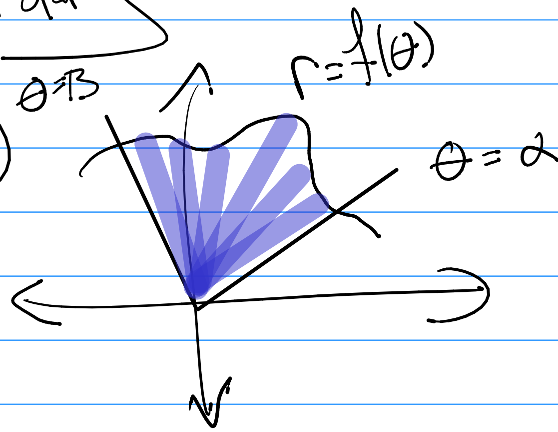
Math 243

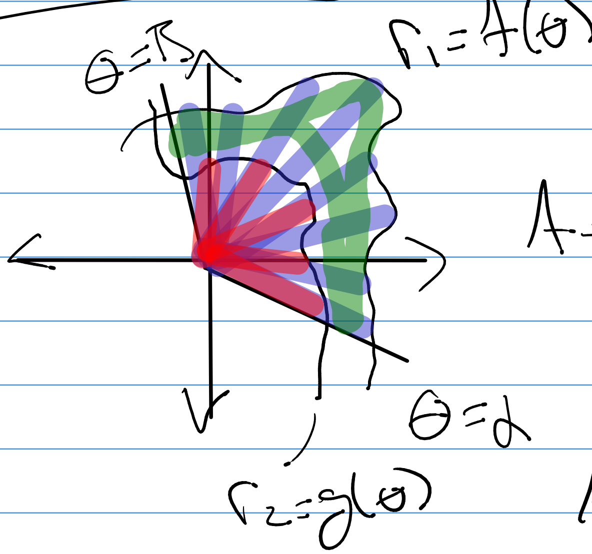
Polar

Area



$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

Area between

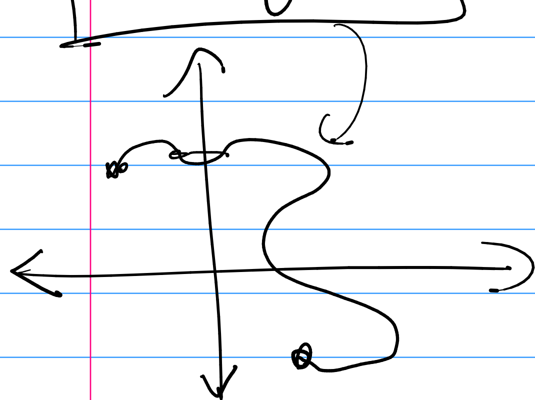


Between = Outer − Inner

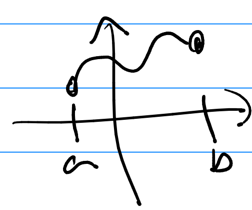
$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_1^2 \, d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_2^2 \, d\theta$$

$$A = \frac{1}{2} \int_{\alpha}^{\beta} (r_1^2 - r_2^2) \, d\theta$$

Arc length



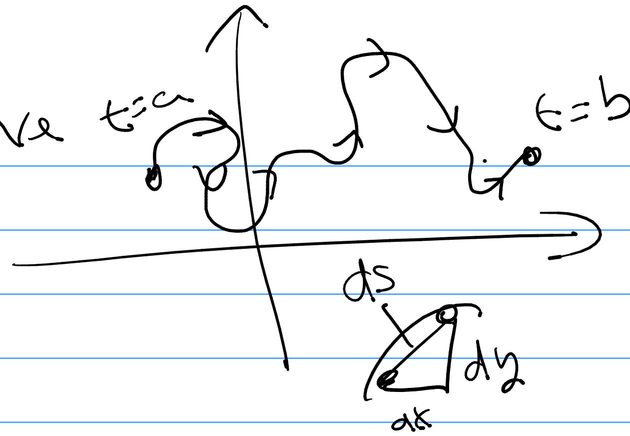
Cartesian: $AL = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$



$$\to \int_a^b \sqrt{dx^2 + dy^2}$$

$$= \int_a^b ds$$

any parametric curve



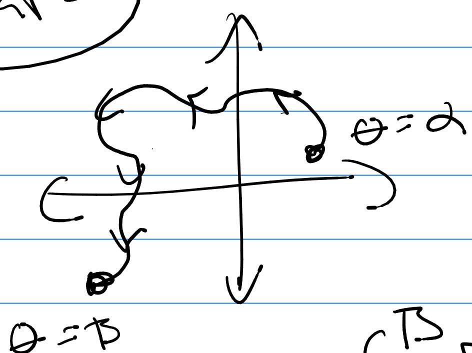
$$AL = \int_a^b ds$$

$$AL = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Polar.

$r = f(\theta)$

$$\boxed{\begin{aligned} x(t) &= r(t)\cos(t) \qquad \alpha \le t \le \beta \\ y(t) &= r(t)\sin(t) \end{aligned}}$$



$$AL = \int_\alpha^\beta \sqrt{(x')^2 + (y')^2}\, dt$$ where $x(t), y(t)$ are

Note: $x = r\cos(t)$ $\qquad$ $y = r\sin t$

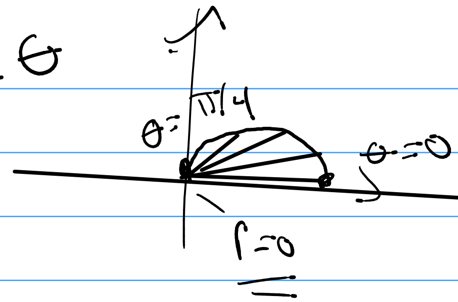
$x' = r'\cos t - r\sin t$ $\qquad$ $y' = r'\sin t + r\cos t$

$$\begin{aligned} (x')^2 &= (r')^2\cos^2 t - \cancel{2rr'\cos t\sin t} + r^2\sin^2 t \\ +\ (y')^2 &\quad\ + (r')^2\sin^2 t + \cancel{2rr'\cos t\sin t} + r^2\cos^2 t \end{aligned}$$

$$(x')^2 + (y')^2 = (r')^2 + r^2$$

So $$\boxed{AL = \int_\alpha^\beta \sqrt{(r')^2 + r^2}\, d\theta}$$

ex one loop of $r = \cos 2\theta$



1. Area
2. Arc length

$$A = 2\int_0^{\pi/4} \frac{1}{2} r^2 \, d\theta$$

$0 = \cos 2\theta$

$2\theta = \pi/2$

$\theta = \pi/4$

$$= \int_0^{\pi/4} \cos^2 2\theta \, d\theta$$

$\cos 2x = 2\cos^2 x - 1$

$$= \frac{1}{2}\int_0^{\pi/4} (\cos 4\theta + 1)\, d\theta$$

$\cos^2 \square = \dfrac{\cos 2\square + 1}{2}$

$$= \frac{1}{2}\left[\frac{1}{4}\sin 4\theta + \theta\right]\Big|_0^{\pi/4} = \frac{1}{2}\left[\pi/4\right] = \boxed{\pi/8}$$

Arc length

$r = \cos(2\theta)$

$r' = -2\sin(2\theta)$

$$AL = \int_\alpha^\beta \sqrt{(r')^2 + (r)^2}\, d\theta$$

$$AL = 2\int_0^{\pi/4} \sqrt{4\sin^2 2\theta + \cos^2 2\theta}\, d\theta \overset{?}{=} 2\int_0^{\pi/4} \sqrt{1 + 3\sin^2 2\theta}\, d\theta$$

idea? #1

#2 ? idea: let $u = \sin^2 2\theta \Rightarrow \cos^2 2\theta = 1 - u$

$du = 2(\sin 2\theta)^1(\cos 2\theta)(2)\, d\theta$

$du = 4\sin 2\theta \cos 2\theta \, d\theta$

$$\frac{1}{2}\int_{\theta=0}^{\theta=\pi/4} \sqrt{\frac{4}{1-u} + \frac{1}{u}}\, du = \frac{1}{2}\int_{\theta=0}^{\theta=\pi/4} \sqrt{\frac{3u+1}{u(1-u)}}\, du$$

? idea? #3

$\int \sqrt{4\sin^2 u + \cos^2 u}\, du$

$= \int \sqrt{4\tan^2 u + 1}\,\cos u\, du$

$= \int \sqrt{4\frac{\sin^2 u}{\cos^2 u} + 1}\,\cos u\, du = \int \sqrt{\frac{4w^2}{1-w^2} + 1}\, dw$

let $w = \sin u$
$dw = \cos u\, du$

$= \int \frac{\sqrt{3w^2+1}}{\sqrt{1-w^2}}\, dw$

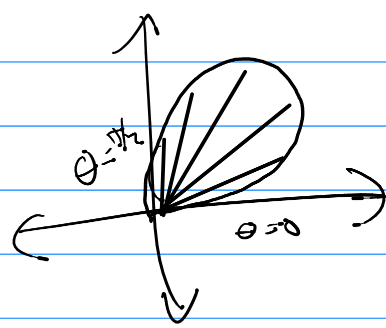
→ Simpson's Rule.

$AL = 2\int_0^{\pi/4} \sqrt{4\sin^2 2\theta + \cos^2 2\theta}\, d\theta \approx \boxed{1.21}105602756$

Ex) $r^2 = \sin 2\theta$ (one petal)

$r = \sqrt{\sin 2\theta}$

$r = -\sqrt{\sin 2\theta}$



$A = \int_0^{\pi/2} \frac{1}{2} r^2\, d\theta = \int_0^{\pi/2} \frac{1}{2} \sin 2\theta\, d\theta$

$= -\left(\frac{1}{2}\right)\frac{1}{2}\cos 2\theta \Big|_0^{\pi/2} = \left(-\frac{1}{4}\cos 2\theta\right)\Big|_0^{\pi/2}$

$= \left(\frac{1}{4}\right) - \left(-\frac{1}{4}\right) = \boxed{\frac{1}{2}}$ units^2

$$AL = \int_0^{\pi/2} \sqrt{(r')^2 + (r)^2}\, d\theta$$

$r^2 = \sin 2\theta$

$2r r' = 2\cos 2\theta$

$r' = \dfrac{\cos 2\theta}{\sqrt{\sin 2\theta}}$

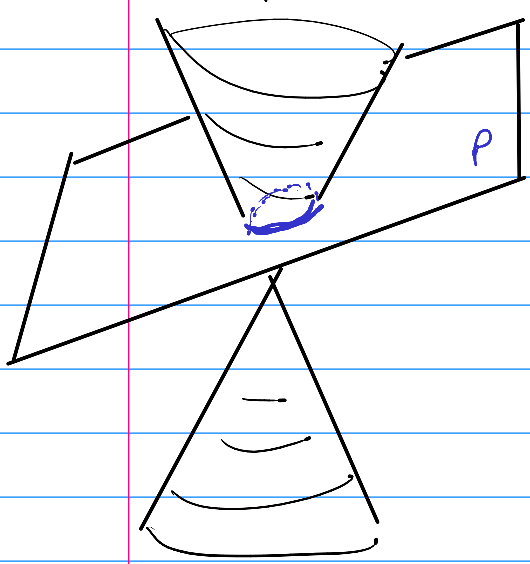
$$AL = \int_0^{\pi/2} \sqrt{\frac{\cos^2 2\theta}{\sin 2\theta} + \sin 2\theta}\, d\theta$$

$$AL = \int_0^{\pi/2} \sqrt{\frac{1}{\sin 2\theta}}\, d\theta = \int_0^{\pi/2} \sqrt{\csc 2\theta}\, d\theta$$

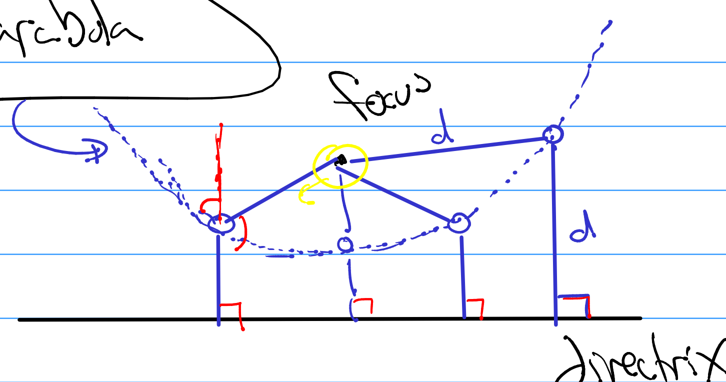
$$AL \approx \boxed{2.6}$$

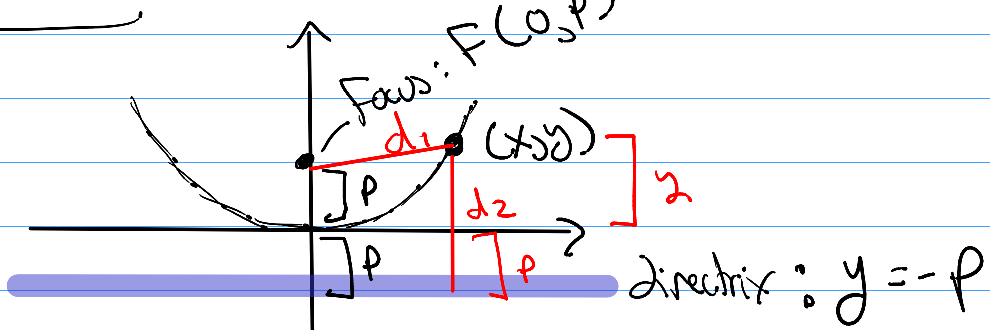
$\boxed{10.5}$ Conic Sections

parabola, ellipse (circle), hyperbola



Parabola





given $d_1 = d_2$ and $d_1^2 = (x-0)^2 + (y-p)^2$

and $d_2 = y + p$

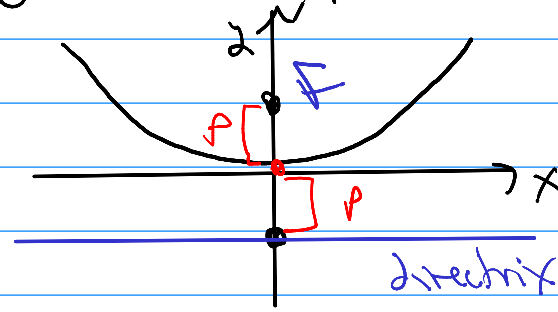
$d_1 = d_2 \rightarrow d_1^2 = d_2^2 \rightarrow x^2 + (y-p)^2 = (y+p)^2$

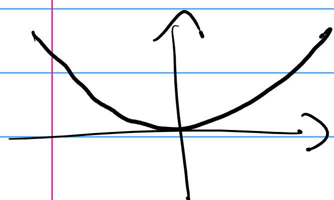
so $x^2 + \cancel{y^2} - 2py + \cancel{p^2} = \cancel{y^2} + 2py + \cancel{p^2}$

$\Rightarrow \boxed{x^2 = 4py}$

or

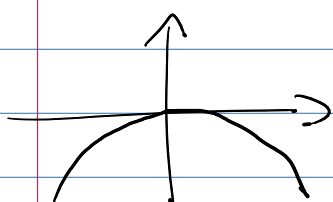
$y = \frac{1}{4p} x^2$



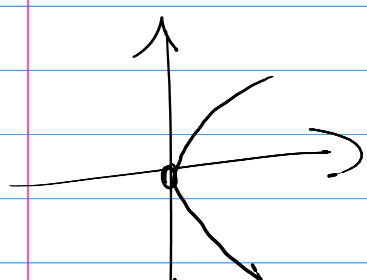


$x^2 = 4py \quad p > 0$

or $y = \frac{1}{4p} x^2$

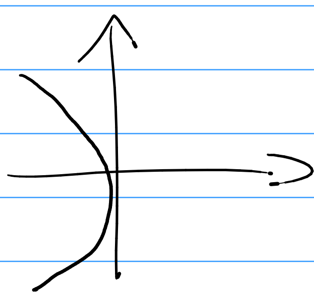


$x^2 = 4py \quad p < 0$

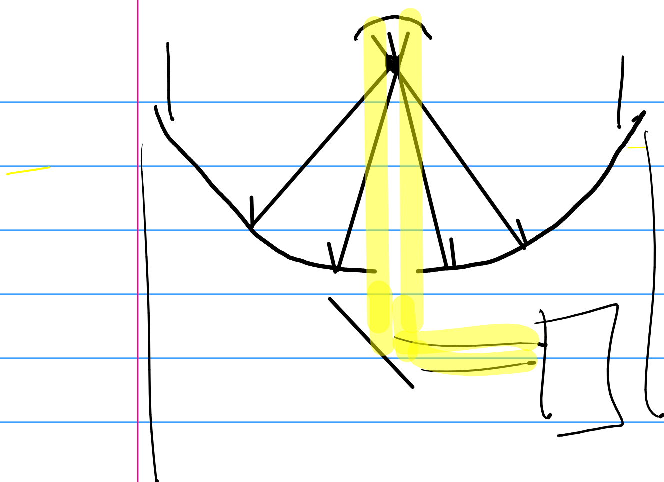


$y^2 = 4px \quad p > 0$

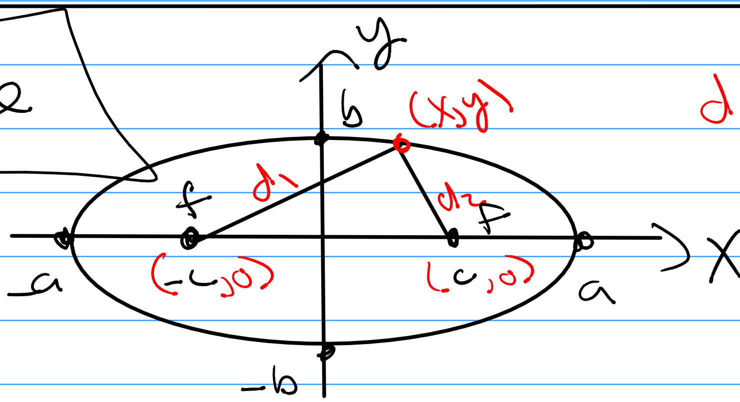
or $x = \frac{1}{4p} y^2$



$y^2 = 4px \quad p < 0$



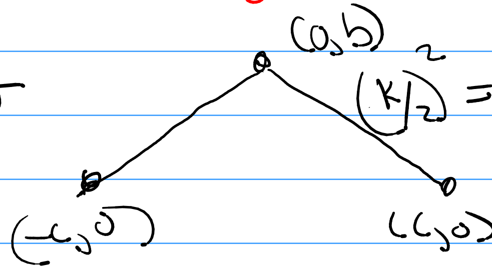
Ellipse



$d_1 + d_2 = \text{constant}$

$d_1^2 = (x+c)^2 + y^2 \qquad d_2^2 = (x-c)^2 + y^2$

K = Constant



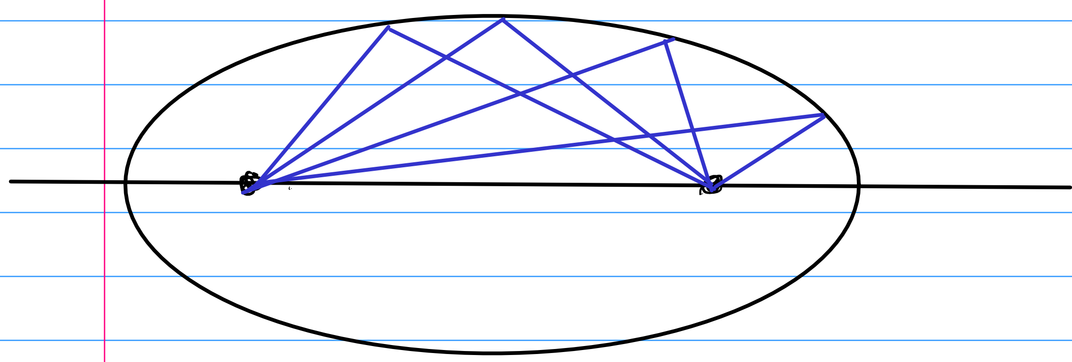
$\left(K/2\right)^2 = c^2 + b^2$

$\Rightarrow K^2 = 4(c^2+b^2)$

algebra

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \underline{\text{and}} \quad c^2 = a^2 - b^2$$

foci $(\pm c, 0)$



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$a > b$	x-axis is main axis
$b > a$	y-axis is main
$a = b$	circle

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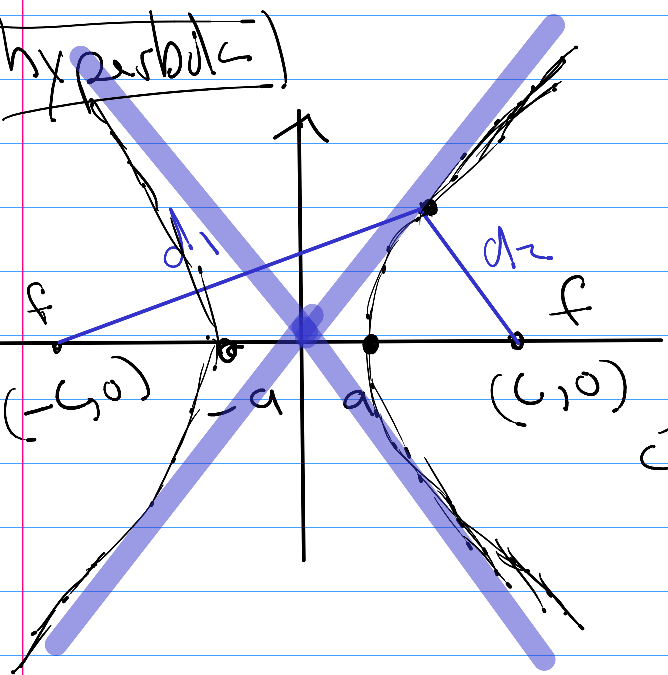
hyperbola

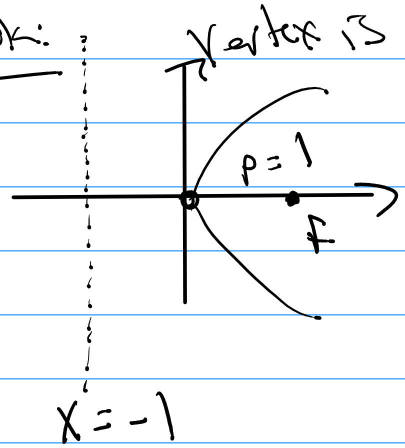


$|d_1 - d_2| =$ constant

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$c^2=a^2+b^2$ asym $= \pm\frac{b}{a}x$

Parabola: vertex is $(0,0)$ focus $(1,0)$

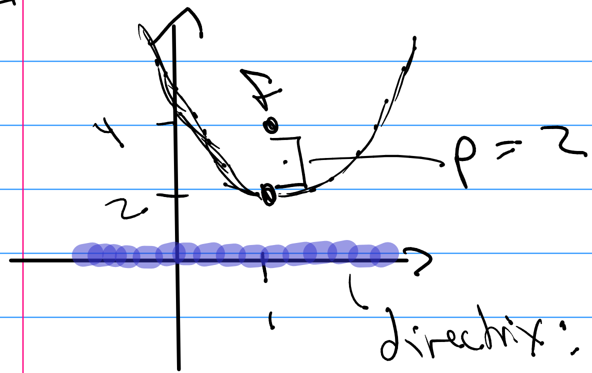


$$x=\frac{1}{4(1)}y^2$$

or

$$4x=y^2$$

Parabola: vertex is $(1,2)$ focus is $(1,4)$



@ origin $y = \frac{1}{4(2)}x^2$

@ $(1,2)$: $\boxed{y = \frac{1}{8}(x-1)^2 + 2}$

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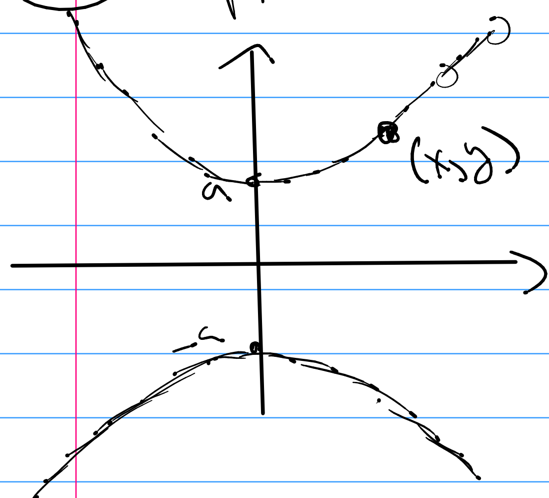
(53) upper branch of $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ is concave up



Find y', y''

$$\frac{d}{dx}\left[\frac{(y)^2}{a^2} - \frac{x^2}{b^2}\right] = \frac{d}{dx}[1]$$

$$\frac{2y}{a^2}y' - \frac{2x}{b^2} = 0$$

So $y' = \frac{a^2}{2y} \cdot \frac{2x}{b^2} \rightarrow y' = \frac{a^2}{b^2} \cdot \frac{x}{y}$

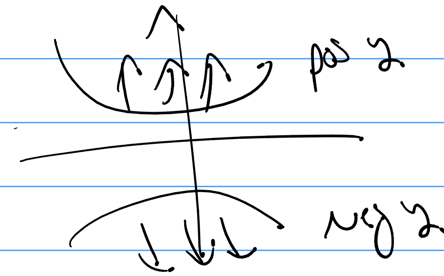
$$y'' = \frac{d}{dx}[y'] = \frac{d}{dx}\left[\frac{a^2}{b^2}\frac{x}{y}\right] = \frac{a^2}{b^2}\frac{d}{dx}\left[\frac{x}{y}\right]$$

$$y'' = \frac{a^2}{b^2}\left[\frac{(1)(y) - (x)(1)(y')}{y^2}\right] = \frac{a^2}{b^2y^2}[y - xy']$$

$$y'' = \frac{a^2}{b^2y^2}\left[y - x\frac{a^2x}{b^2y}\right] = \frac{a^2}{b^2y^3}\left[y^2 - \frac{a^2}{b^2}x^2\right]$$

$$y'' = \frac{a^4}{b^2 y^3}\left[\frac{y^2}{a^2} - \frac{x^2}{b^2}\right]$$

$$y'' = \frac{a^4}{b^2 y^3}$$ so



Q

$x(t) = t\cos t \qquad y(t) = t\sin t$

eqn of line

$\frac{dy}{dx}$ @ $t = \pi$ slope and point (x, y) @ $t = \pi$

$$\boxed{y - y_1 = m(x - x_1)}$$

$x_1 = x(\pi) = \boxed{-\pi}$

$y_1 = y(\pi) = \boxed{0}$

$$m = \left.\frac{dy}{dx}\right|_{t=\pi} = \left.\left[\frac{(1)\cancel{\sin t} + (t)\cos t}{(1)\cos t - (t)\cancel{\sin t}}\right]\right|_{t=\pi} = \boxed{\pi}$$

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \pi(x - (-\pi))$$

$$\boxed{y = \pi(x + \pi)}$$