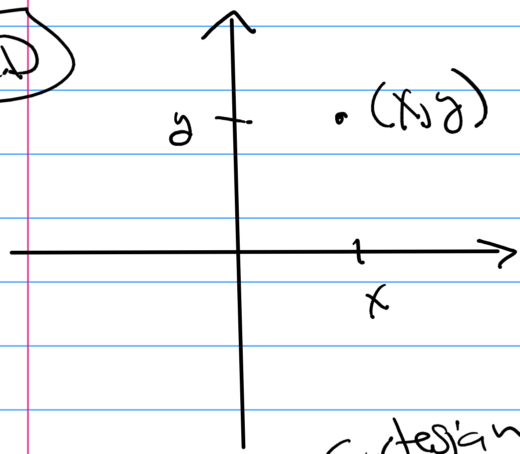
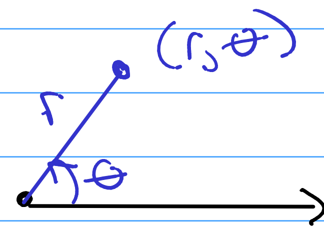
Math 243

Polar Coordinates

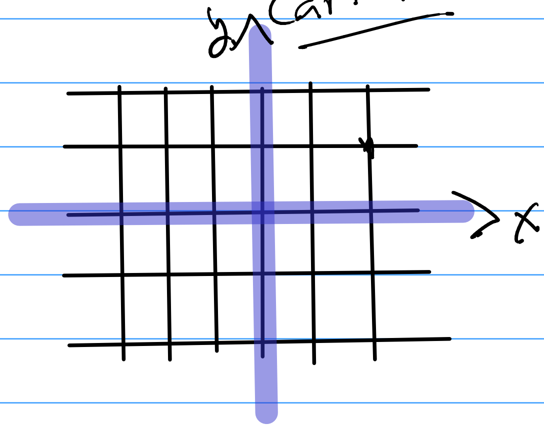
2D

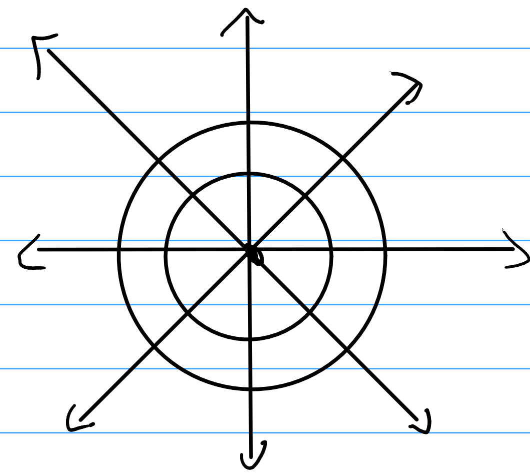




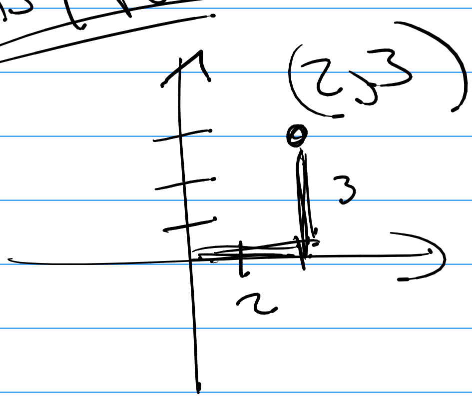
Polar

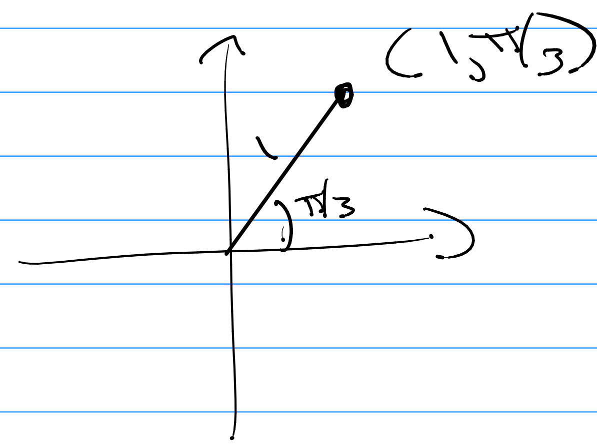
Cartesian



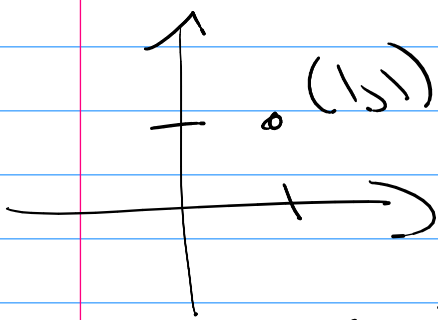


Plots / Points

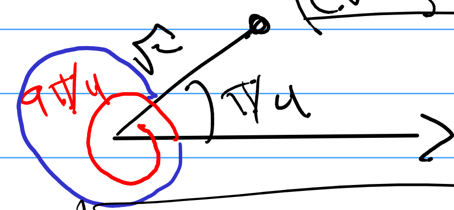




non-uniq. values of (r,θ) for same point



uniq coord.



$0 \le r \qquad 0 \le \theta < 2\pi$

$(\sqrt{2}, \pi/4) = (\sqrt{2}, 9\pi/4)$

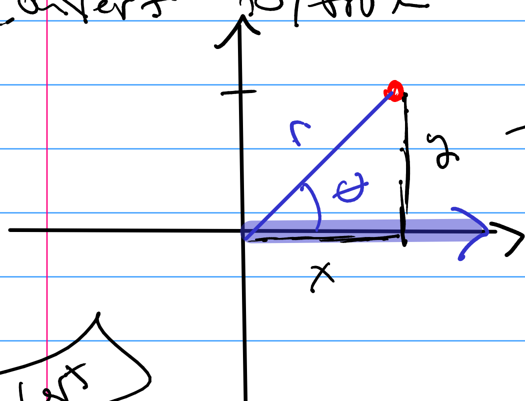
$= (\sqrt{2}, -7\pi/4)$

$= (-\sqrt{2}, 5\pi/4)$

non-uniq coord

if we do not restrict r and θ

Convert to/from $(x,y) \rightleftarrows (r,\theta)$



Facts of this triangle

1. $x^2+y^2=r^2$
2. $\tan\theta = y/x$
3. $\sin\theta = y/r$
4. $\cos\theta = x/r$
5. $\sin^2\theta + \cos^2\theta = 1$
6. all the trig stuff.

Convert

$(r,\theta) \to (x,y)$

want $x = \sim r\text{'s}, \theta\text{'s} \sim$

$y = \sim r\text{'s}, \theta\text{'s} \sim$

(r, θ) In

$$\boxed{\begin{aligned} y &= r\sin\theta \\ x &= r\cos\theta \end{aligned}}$$

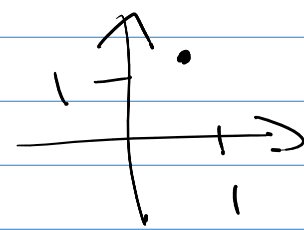
out (x, y)

ex $(r, \theta) = (\sqrt{2}, \pi/4) \longrightarrow (x, y) = (1, 1)$

$$y = \sqrt{2}\sin(\pi/4) = \sqrt{2}\left(\sqrt{2}/2\right) = 1$$
$$x = \sqrt{2}\cos(\pi/4) = \sqrt{2}\left(\sqrt{2}/2\right) = 1$$

ex $(r, \theta) = (1, 1) \rightarrow (x, y) = (\sqrt{2}\cos 1, \sqrt{2}\sin 1)$

$$y = \sqrt{2}\sin(1) \approx 1.19$$
$$x = \sqrt{2}\cos(1) \approx 0.764$$



Convert $(x, y) \rightarrow (r, \theta)$

r = ~ x's and y's ~

θ = ~ x's and y's ~

$$\boxed{\begin{aligned} r^2 &= x^2 + y^2 \\ \tan\theta &= y/x \end{aligned}}$$

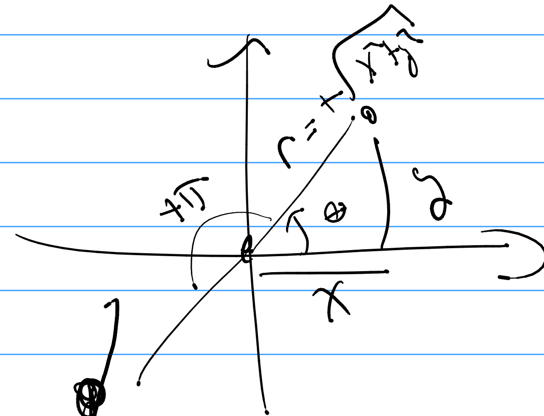
$\rightarrow r = \pm\sqrt{x^2+y^2}$

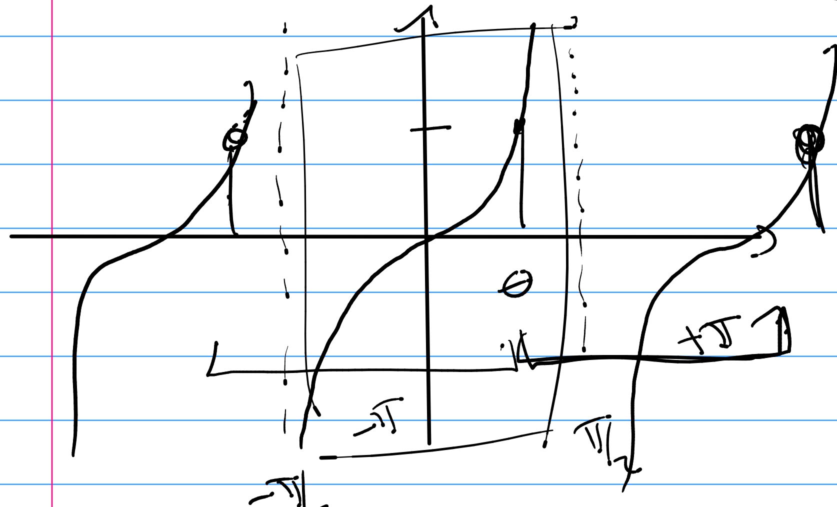
$\rightarrow \theta = \tan^{-1}(y/x)$

(x,y) In $\rightarrow$ $\boxed{\begin{array}{c} r^2 = x^2 + y^2 \\ \tan\theta = y/x \end{array}}$ $\rightarrow$ out (r, θ)

$r = \pm\sqrt{x^2+y^2}$

$\theta = \tan^{-1}(y/x)$

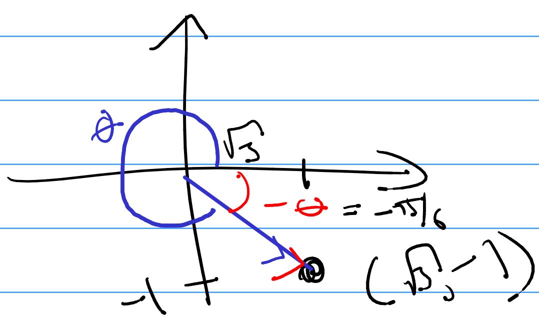




$\boxed{\text{ex}}$ $(x,y) = (\sqrt{3}, -1)$

$r^2 = (\sqrt{3})^2 + (-1)^2 = 4$

$\boxed{r = 2}$



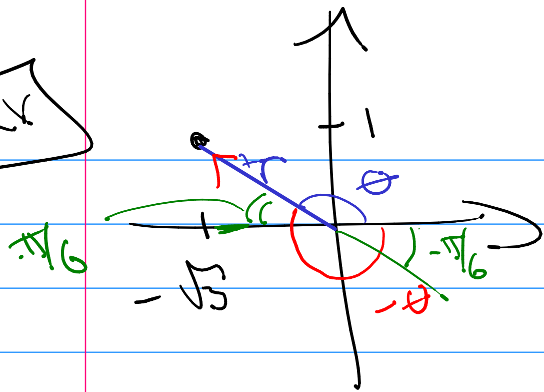
$\tan\theta = -1/\sqrt{3}$

$\theta = \tan^{-1}(-1/\sqrt{3})$

$\theta = -\pi/6$

$\boxed{(2, -\pi/6)} = \boxed{(2, 11\pi/6)}$

ex



$(x,y) = (-\sqrt{3}, 1)$

$r^2 = (-\sqrt{3})^2 + (1)^2 = 4$

$\boxed{r = -2}$

$\tan\theta = \frac{1}{-\sqrt{3}}$

$\theta = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\pi/6$

$(-2, -\pi/6)$

$= \boxed{(2, 5\pi/6)}$

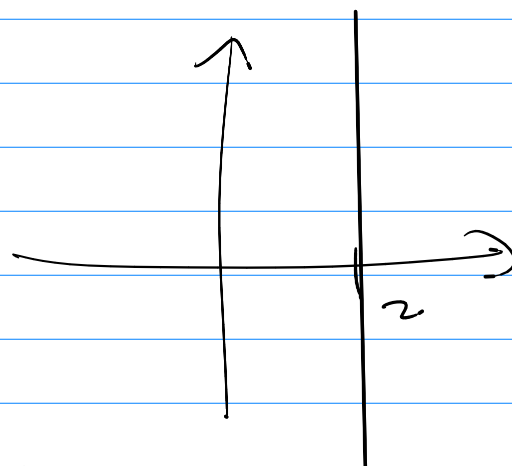
$= (2, 7\pi/6)$

Plots

Cartesian

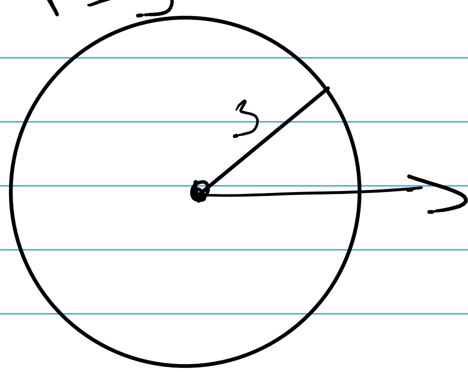
$x = 2$

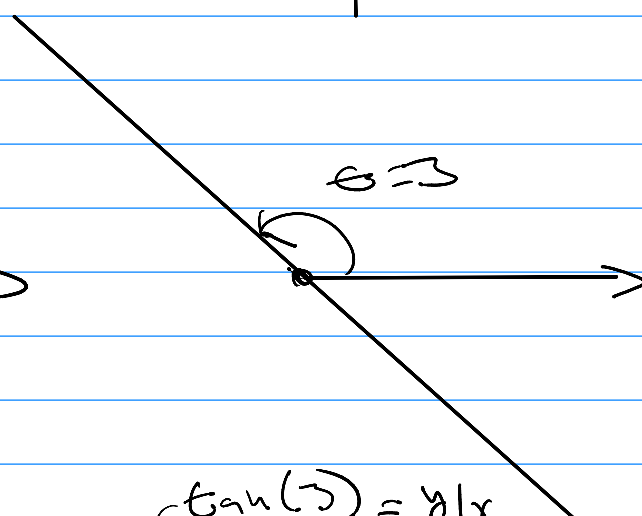
x	y
2	−1
2	0
2	1



Polar

$r = 3$





circle? yes

$\sqrt{x^2+y^2} = 3$

$x^2 + y^2 = (3)^2$

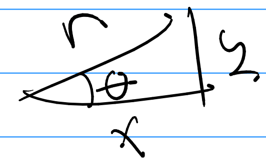
$\tan(3) = y/x$

$y = \tan(3) \cdot x$

line? yes

ex

$\boxed{r^2 \sin 2\theta = 1} \to \boxed{\text{Cartesian?}}$



Know: $\boxed{x^2+y^2=r^2}$

$\to (x^2+y^2)\sin 2\theta = 1$

$\sin\theta = y/r$ $\quad \cos\theta = x/r$

Know: $\boxed{\sin 2\theta = 2\sin\theta\cos\theta}$

$\to \sin 2\theta = 2\,\frac{y}{r}\,\frac{x}{r} = \frac{2yx}{r^2}$

$\to \cancel{r^2}\,\frac{2yx}{\cancel{r^2}} = 1 \to \boxed{y = \frac{1}{2x}}$

Plot $\quad y = \frac{1}{2x}$ is easy

implicit $\quad r^2 \sin 2\theta = 1$

explicit $\quad r^2 = \frac{1}{\sin 2\theta} = \csc(2\theta)$

$r = \pm\sqrt{1/\sin 2\theta} = \pm\sqrt{\csc 2\theta}$

θ	r
0	$\pm\infty$
$\pi/4$	± 1
$\pi/2$	$\pm\infty$

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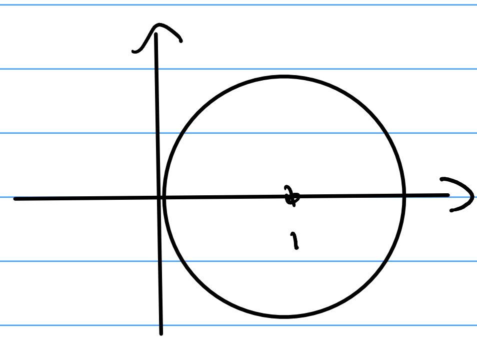
ex $\quad x^2 + y^2 = 2x$

$x^2 - 2x + 1^2 + y^2 = 0 + 1^2$

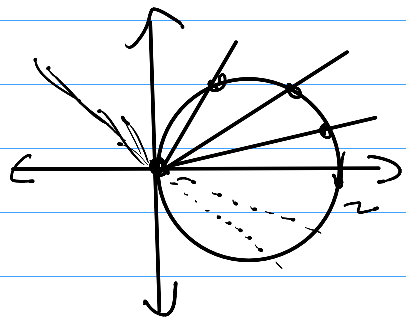
$(x-1)^2 + y^2 = 1$



$x^2 + y^2 = 2x$ in polar?

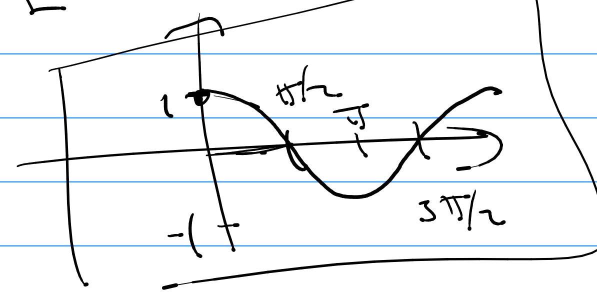
$\to r^2 = 2x$

$\to r^2 = 2(r\cos\theta) \rightsquigarrow \boxed{r = 2\cos\theta}$



θ	r
0	2
$\pi/2$	0
π	-2

$r = 2\cos\theta$ in cartesian



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a) Circle of radius 5 and center $(2,3)$

Cartesian: $r = 5 \quad (h,k) = (2,3)$

$$\boxed{(x-2)^2 + (y-3)^2 = 5^2}$$

Polar $(r\cos\theta - 2)^2 + (r\sin\theta - 3)^2 = 5^2$

$$r^2\cos^2\theta - 4r\cos\theta + 4 + r^2\sin^2\theta - 6r\sin\theta + 9 = 25$$

$$\boxed{r^2 - 4r\cos\theta - 6r\sin\theta = 12}$$

b) Circle @ origin with radius 4

Cartesian $x^2 + y^2 = 4^2$

polar $r = 4$

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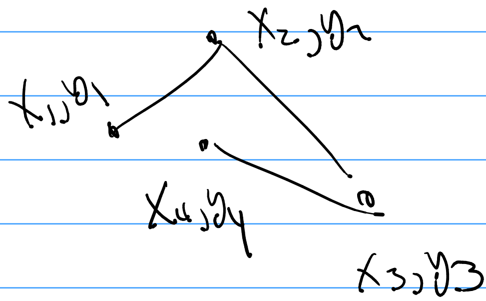
plot — plot x's, y's



x	y
x_1	y_1
x_2	y_2
x_3	y_3
x_4	y_4

Polar (r, θ) given $\rightsquigarrow$ plot system is x's, y's

→ parametric plot

① polar: explicit function $\boxed{r = f(\theta)}$

② $x = r\cos\theta$ $y = r\sin\theta$ polar

③ plot

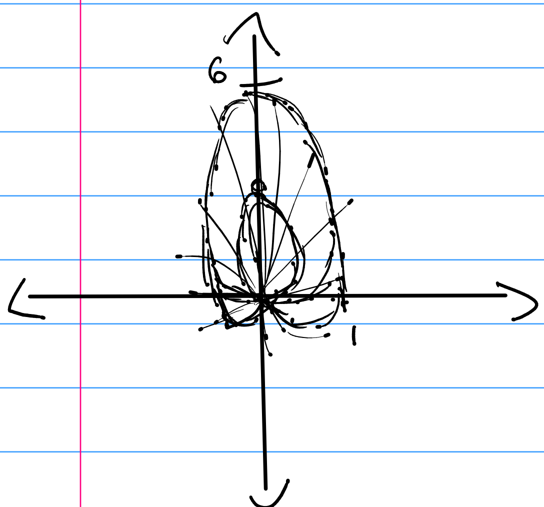
$$\boxed{\begin{aligned} x(t) &= f(t)\cos(t) \\ y(t) &= f(t)\sin(t) \\ \alpha \le\ & t \le \beta \end{aligned}}$$

Ex $r = 1+3\sin\theta$

plot: $x(t) = (1+3\sin t)\cos t$

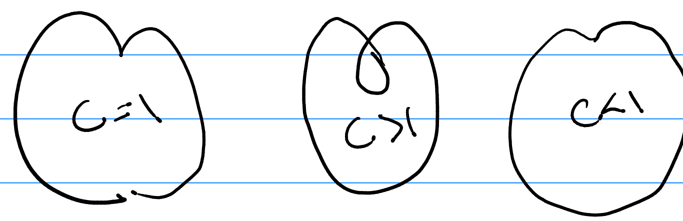
$y(t) = (1+3\sin t)\sin t$

$0 \le t \le 2\pi$



$r = 1 \pm c\sin t$

$r = 1 \pm c\cos t$

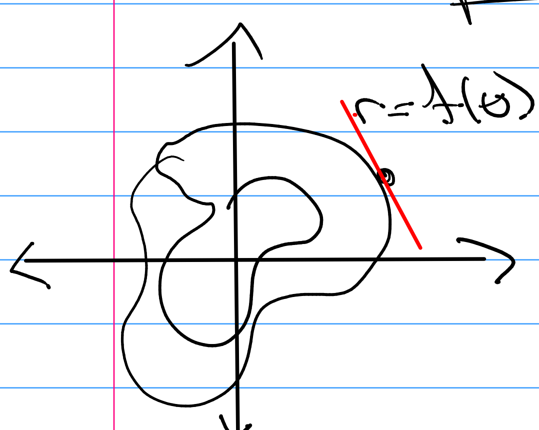


See video for graphing

Calculus $r = f(\theta)$ polar

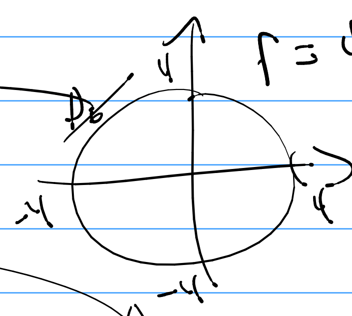
$\rightarrow$ parametric: $y(t) = r\sin(t)$

$x(t) = r\cos(t)$



tangent $= \dfrac{dy}{dx}$

(2) $\dfrac{dr}{d\theta}$



Ex $\dfrac{dr}{d\theta} = 0$

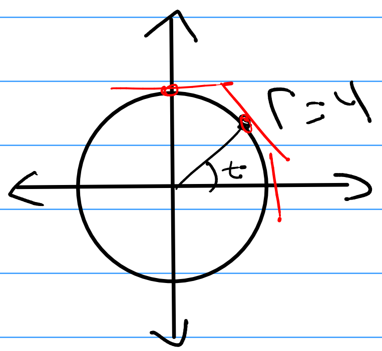
So, tangent lines are still dy/dx

given: $y(t) = r\sin(t)$
$x(t) = r\cos(t)$

polar eqns

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$

ex



$y(t) = 4\sin(t)$
$x(t) = 4\cos(t)$

$$\frac{dy}{dx} = \frac{4\cos(t)}{-4\sin(t)} = -\frac{\cos t}{\sin t}$$

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ex $r = \sin(3\theta)$

$y(t) = \sin(3t)\sin(t)$

$x(t) = \sin(3t)\cos(t)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\cos(3t)\sin(t) + \sin(3t)\cos t}{3\cos(3t)\cos t - \sin(3t)\sin t}$$

horz tangent $3\cos(3t)\sin t + \sin(3t)\cos t = 0$

Vert tangent $3\cos(3t)\cos t - \sin(3t)\sin t = 0$

Solve?

$0 \le t \le \pi$

Solve?:

$$3\cos(3t)\sin t + \sin(3t)\cos t = 0$$

Idea: $\left[\begin{array}{l} \sin(x+y) = \sin x \cos y + \cos x \sin y \\ \cos(x+y) = \cos x \cos y - \sin x \sin y \end{array}\right]$ use?

Idea: $3\cos(3t)\sin t = -\sin(3t)\cos t$?

$$-3\frac{\cos 3t}{\sin 3t} = \frac{\cos t}{\sin t}$$

$$-3\tan 3t = \tan t$$

stuck?: $\rightarrow$ $\boxed{\tan t - 3\tan 3t = 0}$

$$f(t) = \tan t - 3\tan 3t$$

Ex slope of $r = 2\cos\theta$ @ $\theta = \pi/3$

$y(t) = 2\cos(t)\sin(t) = \sin(2t)$

$x(t) = 2\cos(t)\cos(t) = 2\cos^2(t)$

$$\frac{dy}{dx} = -\frac{2\cos(2t)}{4\cos(t)\sin t} = -\frac{1}{2}\frac{\cos 2t}{\cos(t)\sin(t)} = -\cot(2t)$$

$$\left.\frac{dy}{dx}\right|_{t=\pi/3} = \boxed{-\cot\left(2\pi/3\right)}$$

More Calculus

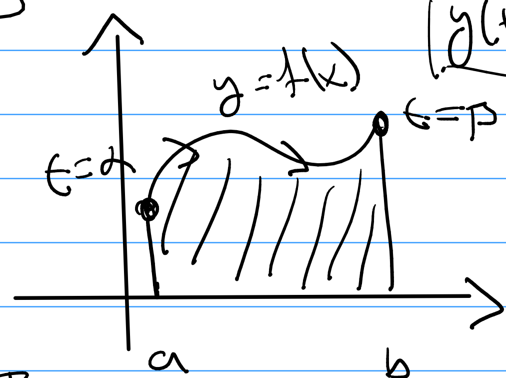
→ Tangents $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

→ **Areas** ?

Parametric

$x(t) \to dx = \boxed{x'(t)\,dt}$

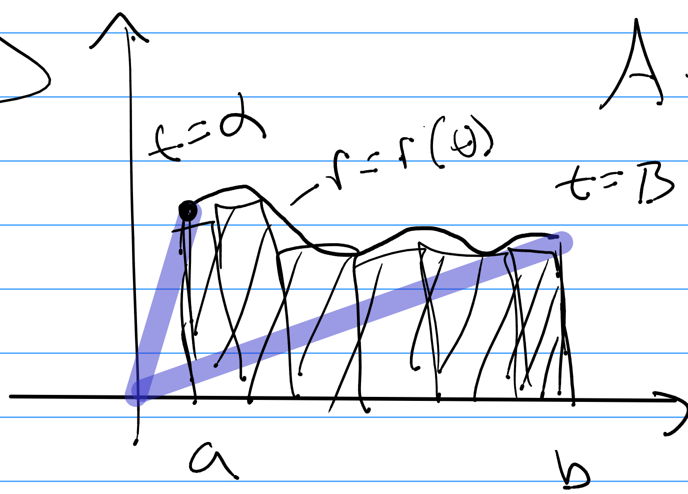
$\boxed{y(t)}$



$$A = \int_{\alpha}^{\beta} \boxed{y}\, dx$$

$$A = \int_{\alpha}^{\beta} y(t)\, x'(t)\, dt$$

Similar

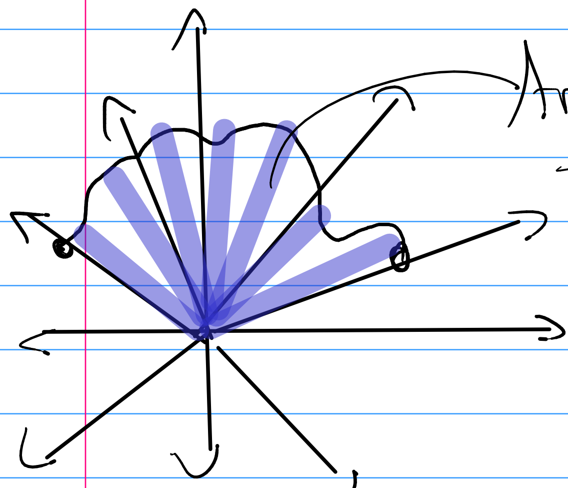


$$A = \int_{\alpha}^{\beta} y(t)\, x'(t)\, dt$$

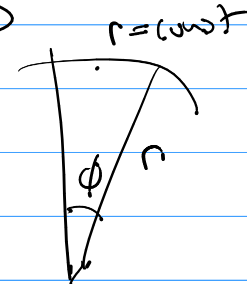
$y(t) = r(t)\sin(t)$

$x(t) = r(t)\cos(t)$

Typically Area for a polar curve is not the above, but ... area swept out.



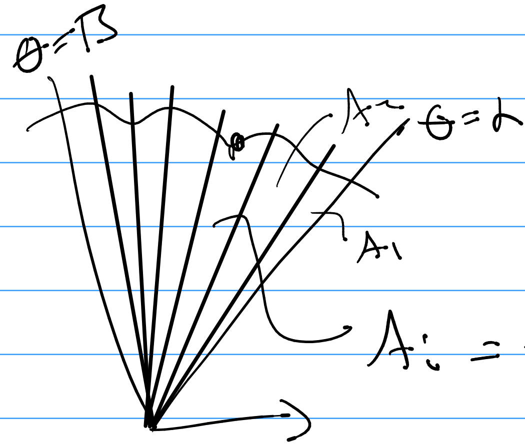
Area swept by the curve?



$\approx$ part of a circle

$$A = \tfrac{1}{2}\phi r^2$$

Idea



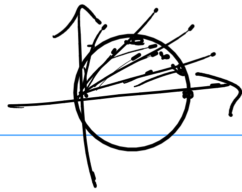
$$A_i = \tfrac{1}{2}(\Delta\theta)\left[f(\theta_i^*)\right]^2$$

$$A \approx \sum_{i=1}^{n} A_i \quad \text{and} \quad \lim_{n\to\infty} \sum_{i=1}^{n} A_i = A$$

$$\text{so } A = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\tfrac{1}{2}\left[f(\theta_i^*)\right]^2}_{\text{integrand}} \, \underbrace{\boxed{\Delta\theta}}_{d\theta}$$

$$\therefore \boxed{A = \int_{\theta=\alpha}^{\theta=\beta} \tfrac{1}{2} r^2 \, d\theta}$$

ex



$r = 2\cos\theta$

area of circle $= 2\int_0^{\pi/2} \frac{1}{2} r^2 \, d\theta$

$$= 4\int_0^{\pi/2} \cos^2\theta \, d\theta = 2\int_0^{\pi/2} \cos 2\theta + 1 \, d\theta$$

$$= 2\left[\frac{1}{2}\cancel{\sin 2\theta} + \theta\right]\Big|_0^{\pi/2}$$

$$= 2(\pi/2) = \boxed{\pi}$$