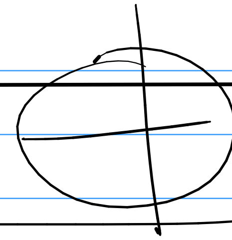
Math 243

Q's

$$\boxed{x^2+y^2=r^2}$$



$$\boxed{x(t)=r\cos t \qquad y(t)=r\sin(t) \qquad 0\le t\le 2\pi}$$

check: $(r\cos t)^2+(r\sin t)^2$

$= r^2\cos^2 t + r^2\sin^2 t$

$= r^2(\cos^2 t+\sin^2 t)$

$= r^2$

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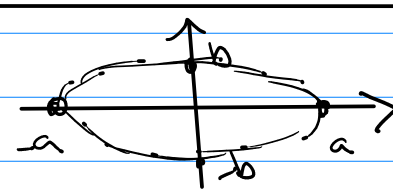
ex ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

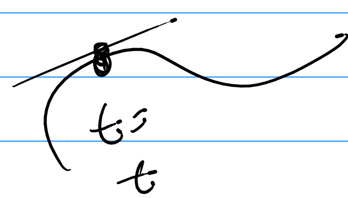


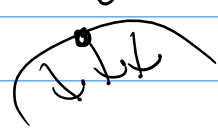
$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$$

$x(t)=a\cos t \qquad y(t)=b\sin t$

check: $\left(\dfrac{\cancel{a}\cos t}{\cancel{a}}\right)^2+\left(\dfrac{\cancel{b}\sin t}{\cancel{b}}\right)^2=1$ ✓

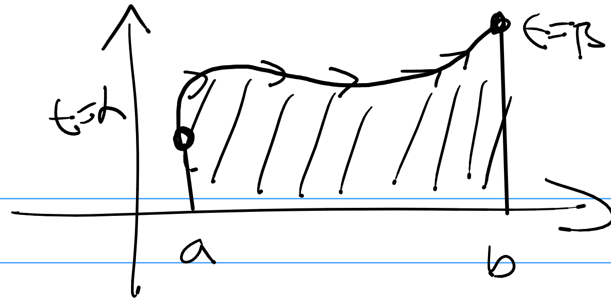
$\boxed{10.2}$ Calculus on $x(t), y(t)$ $\alpha\le t\le\beta$ parametric curves.

①  slope of tangent line?

②  curvature?

Calculus

③ Area



$y = f(x)$

$x(t)$

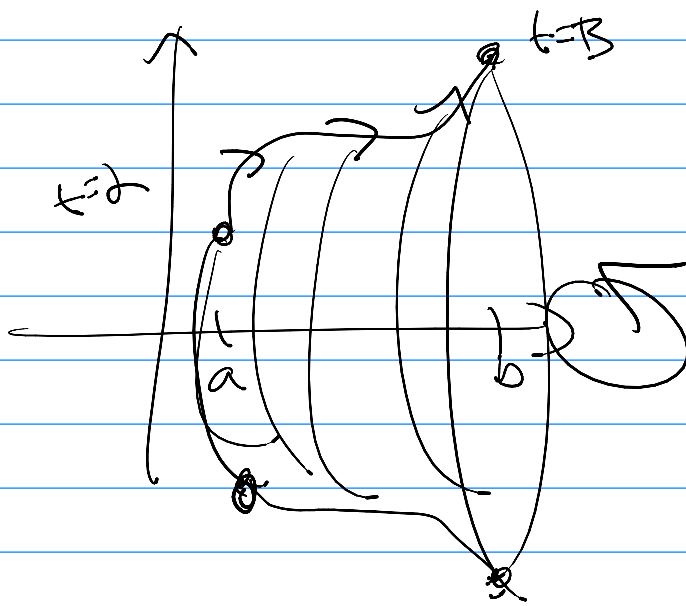
$y(t)$

④ Arclength

$y = f(x)$

$\begin{cases} x(t) \\ y(t) \end{cases}$

⑤ Surface Area of revolution



$y = f(x)$

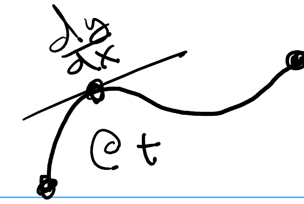
$\begin{cases} x(t) \\ y(t) \end{cases}$

Derivatives: ① $y = f(x)$ $\quad \frac{dy}{dx}$ we can find this.

② $y = f(g(x))$ $\quad \frac{dy}{dx} = f'(g(x))\, g'(x)$

chain rule.

parametric curve (find $\frac{dy}{dx}$?)

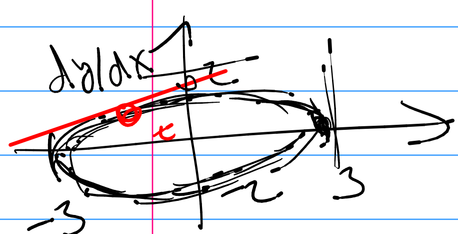


have $x(t)$, $y(t)$, know $y = f(x)$

$\rightarrow$ $y(t) = f(x(t))$

by chain rule $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$ $\rightsquigarrow$ $\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad dx/dt \neq 0}$

(ex) $x(t) = 3\cos t$ $\quad$ $y(t) = 2\sin t$



$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{-3\sin t} = -\frac{2}{3}\cot t$$

Curvature

2^{nd} Derivative $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]$ ← function

Calc 1 $\quad y = x^3 + \sin x$

$$\frac{dy}{dx} = 3x^2 + \cos x$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = 6x - \sin x$$

Calc 2 with parametric eq'n's? $\quad$ $x(t), y(t)$?

still $\frac{d}{dx}\left[\frac{dy}{dx}\right]$

slope function

and $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ says $\frac{d}{dx}[y] = \frac{\frac{d}{dt}[y]}{\frac{d}{dt}[x]}$

any function of t

this would give.. $z(t) = t^3 + \cos t$

$x(t) = 2t^2$

ex $\frac{d}{dx}[z] = \frac{\frac{d}{dt}[z]}{\frac{d}{dt}[x]} = \frac{3t^2 - \sin t}{4t}$

ex $\frac{d}{dx}[\tan t + \sin(3t)] = \frac{\frac{d}{dt}[\tan t + \sin 3t]}{\frac{d}{dt}[x]} = \frac{\sec^2 t + 3\cos 3t}{4t}$

back to 2nd derivative.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{d}{dt}[x]}$$

ex $x(t) = 3\cos t \quad y(t) = 2\sin t$

slope: $\frac{d}{dx}[y] = \frac{\frac{d}{dt}[y]}{\frac{d}{dt}[x]} = \frac{2\cos t}{-3\sin t} = -\frac{2}{3}\cot t$

Curvature: $\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[-\frac{2}{3}\cot t\right]$

$$= \frac{\frac{d}{dt}\left[-\frac{2}{3}\cot t\right]}{\frac{d}{dt}\left[3\cos t\right]} = \frac{\frac{2}{3}\csc^2 t}{-3\sin t} = -\frac{2}{9}\csc^3 t$$

3rd Derivative

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = \frac{d}{dx}\left[-\frac{2}{9}\csc^3 t\right]$$

$$= \frac{\frac{d}{dt}\left[-\frac{2}{9}\csc^3 t\right]}{\frac{d}{dt}\left[3\cos t\right]} = \frac{\frac{2}{3}\csc^3 t\cot t}{-3\sin t} = \boxed{-\frac{2}{9}\csc^4 t\cot t}$$

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ex $x = t^3 - 3t \qquad y = t^2 - 3$ horz or vert. tangents

Max? Min?

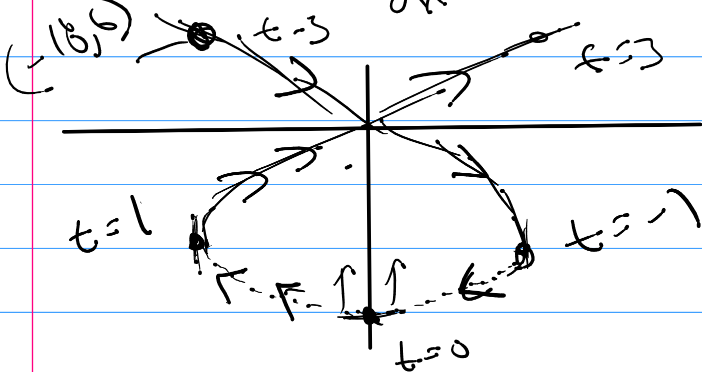
$$\frac{dy}{dx} = \frac{2t}{3t^2-3}$$

tangents

$\to$ horz $\frac{dy}{dx} = 0 \to 2t = 0 \to t = 0 \xrightarrow{pt} (0,-3)$

$\to$ Vert $\frac{dy}{dx}$ dne $\to 3t^2 - 3 = 0 \to t = +1 \to (-2,-2)$

$-1 \to (2,-2)$



t	x	y
-3	-18	6
-2		
2		
3		

Curvature

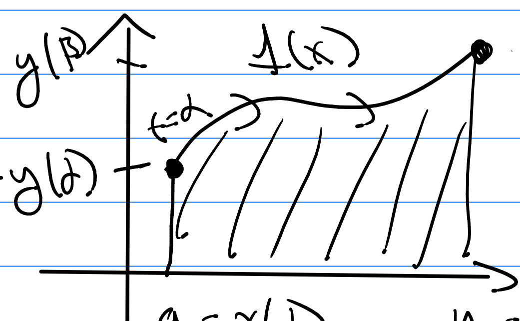
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] \quad \text{and} \quad \frac{dy}{dx} = \frac{2t}{3t^2-3}$$

$$\text{So } \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[2t/3t^2-3\right]}{\frac{d}{dt}\left[t^3-3t\right]} = \frac{\left[\frac{2(3t^2-3)-(2t)(6t)}{(3t^2-3)^2}\right]}{\frac{(3t^2-3)}{1}}$$

$$\text{So } \frac{d^2y}{dx^2} = \frac{2(3t^2-3)-12t^2}{(3t^2-3)^3}$$

$$= \frac{-6t^2-6}{3^3(t^2-1)^3} = \frac{-2(t^2+1)}{9(t^2-1)^3}$$

Area



$(x(t), y(t))$ on $\alpha \le t \le \beta$

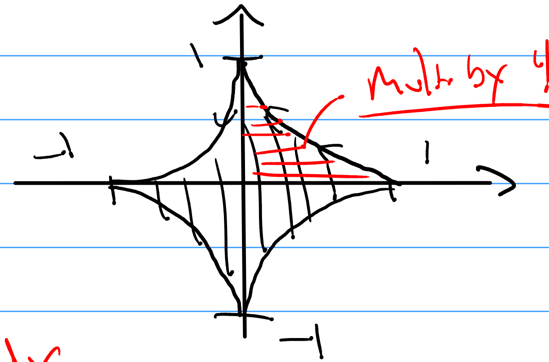
$$\text{Area} = \int_{a=x(\alpha)}^{b=x(\beta)} y\,dx = \boxed{\int_\alpha^\beta y(t)\,x'(t)\,dt}$$

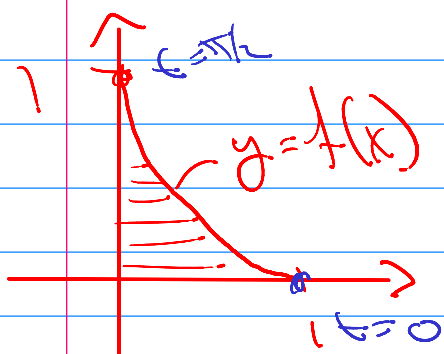
$y = y(t) \qquad x = x(t)$

$dx = x'(t)\,dt$

ex $x = \cos^3 t \quad y = \sin^3 t$ astroid

$0 \le t \le 2\pi$





$$\int_0^1 f(x)\,dx$$

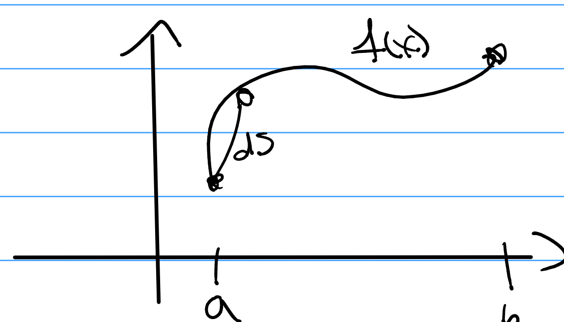
$$\text{Area} = 4\int_{\pi/2}^{0} \underbrace{(\sin^3 t)}_{y}\,\underbrace{(3\cos^2 t)(-\sin t)\,dt}_{dx}$$

$$\text{Area} = 12\int_0^{\pi/2} \sin^4 t \cos^2 t\,dt = 12\int_0^{\pi/2} \sin^4 t(1-\sin^2 t)\,dt$$

$$\text{Area} = 12\int_0^{\pi/2} (\sin^4 t - \sin^6 t)\,dt$$

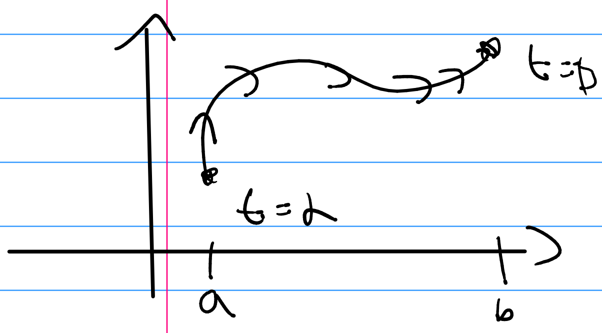
$$= \boxed{\frac{3\pi}{8}}$$

Arc length



$$\int_a^b ds$$

$$AL = \int_a^b \sqrt{1+(f')^2}\,dx$$



$x(t)$
$y(t)$ on $\alpha \le t \le \beta$

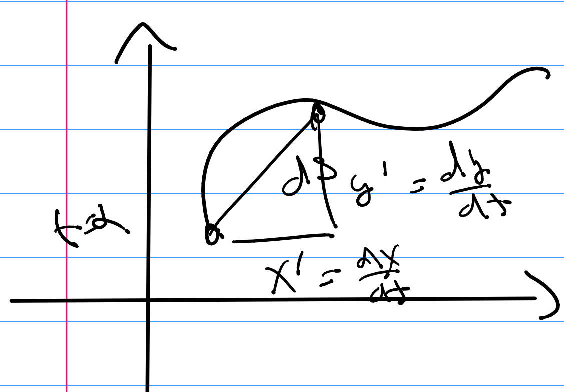
$AL = ?$

<u>Know:</u> $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ and $x = x(t)$
$dx = \frac{dx}{dt}\,dt$

$$AL = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$AL = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\,\frac{dx}{dt}\,dt$$

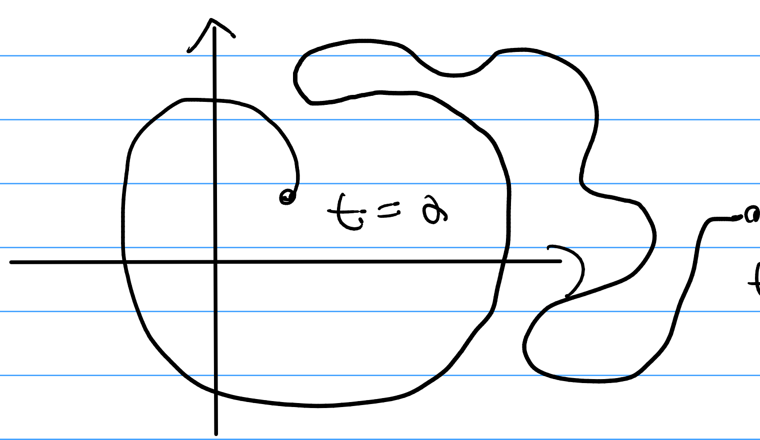
$$\boxed{AL = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt}$$



is just $\int_\alpha^\beta ds$

→ formula works for non-explicit $y = f(x)$

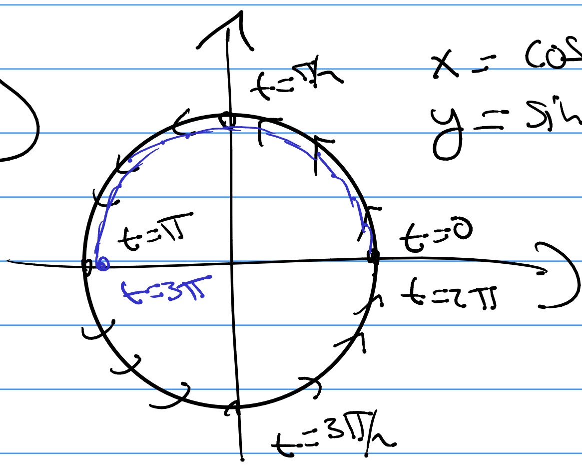
ex



$$AL = \int_\alpha^\beta \sqrt{(x')^2 + (y')^2}\,dt$$

but if the curve re-traces itself AL continues to grow --

ex



$x = \cos t$
$y = \sin t$

$$AL = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt$$
$$= \int_0^{2\pi} 1\, dt = t\Big|_0^{2\pi} = \boxed{2\pi}$$

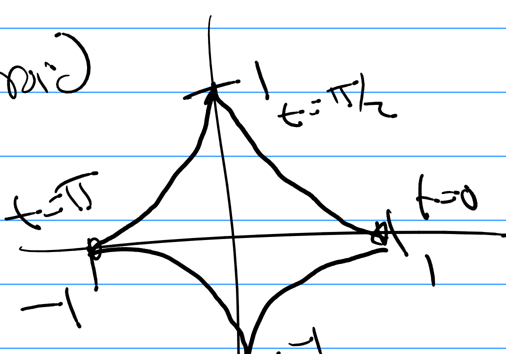
$$AL = \int_0^{3\pi} 1\, dt = \boxed{3\pi}$$

Q $L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

① length of a curve, C, that is traversed once?
(find α, β that map C once)

② distance between $t=\alpha$, $t=\beta$?
(allow multiple traversals)

ex astroid



$x = \cos^3 t$
$y = \sin^3 t$

$$L = \int_0^{2\pi} \left(\left(-3\cos^2 t \sin t\right)^2 + \left(3\sin^2 t \cos t\right)^2\right)^{1/2} dt$$

$$L = 3\int_0^{2\pi} \left(\cos^4 t \sin^2 t + \sin^4 t \cos^2 t\right)^{1/2} dt$$

$$\left(\cos^2 t \sin^2 t \left[\cos^2 t + \sin^2 t\right]\right)^{1/2}$$

$$\sqrt{\cos^2 t}\sqrt{\sin^2 t}\sqrt{\cos^2 t + \sin^2 t}$$

$$L = 12\int_0^{\pi/2} |\sin t||\cos t|\sqrt{\cos^2 t + \sin^2 t}\, dt$$

$$L = 12\int_0^{\pi/2} \sin t \cos t \, dt = 12\int_0^1 u\, du$$

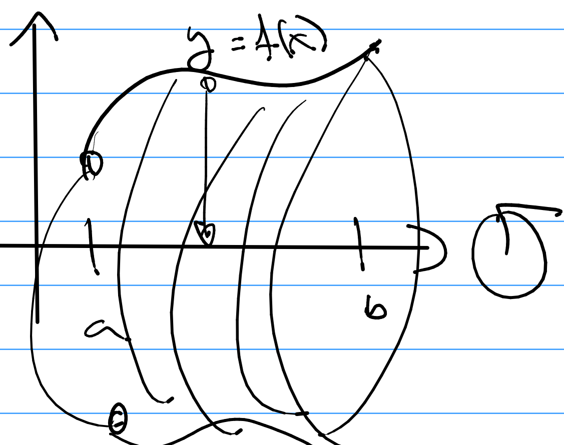
$u = \sin t$

$du = \cos t\, dt$

$$= 6u^2\Big|_0^1 = \boxed{6}$$

Surface Area

$y = f(x)$



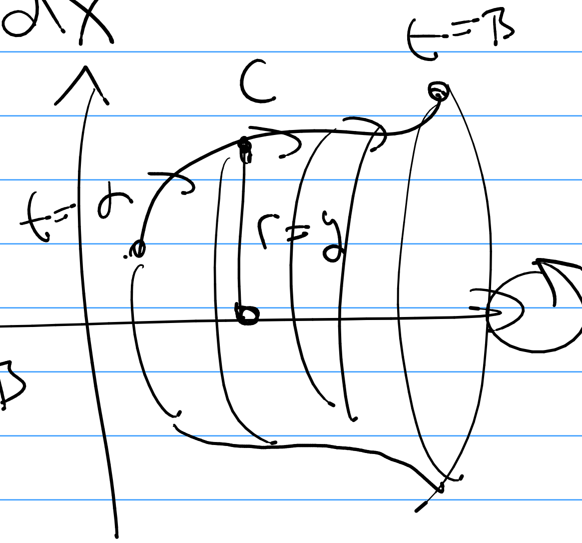
$$SA = \int_a^b 2\pi f(x)\sqrt{1 + (f')^2}\, dx$$

$$SA = \int_\alpha^\beta 2\pi y(t)\sqrt{(x')^2 + (y')^2}\, dt$$

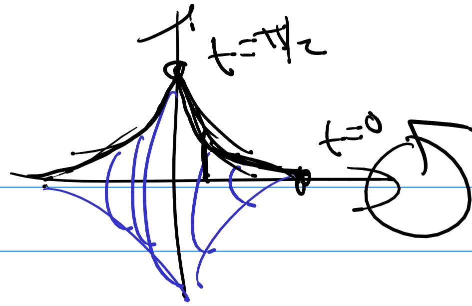
$x(t)$

$y(t)$

$\alpha \le t \le \beta$



$\boxed{\text{astroid}}$



$x(t) = \cos^3 t$

$y(t) = \sin^3 t$

$$SA = 2\int_0^{\pi/2} 2\pi(\sin^3 t)\sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2}\, dt$$

$$SA = 12\pi \int_0^{\pi/2} \sin^3 t \sin t \cos t \, dt$$

$$SA = 12\pi \int_0^{\pi/2} \sin^4 t \cos t \, dt = 12\pi \int_0^1 u^4 \, du$$

let $u = \sin t$

$du = \cos t \, dt$

$$= \boxed{\frac{12}{5}\pi}$$

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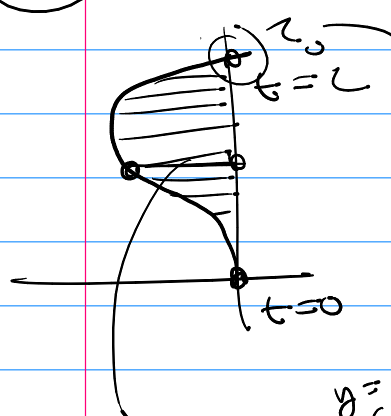
ex Area inside $x = t^2 - 2t$, $y = \sqrt{t}$ and y-axis

$dy = \frac{1}{2} t^{-1/2} \, dt$ $\qquad x = 0$



$0 = t^2 - 2t$

$0 = t(t-2)$

$t = 0, \; t = 2$

$$\text{Area} = \int_{y=0}^{y=\sqrt{2}} x \, dy = \int_{t=0}^{t=2} (t^2 - 2t)\frac{1}{2} t^{-1/2} \, dt$$

$$\text{Area} = \frac{1}{2}\int_0^2 t^{3/2} - 2t^{1/2} \, dt = \frac{1}{2}\left[\frac{2}{5}t^{5/2} - \frac{4}{3}t^{3/2}\right]_0^2$$

$$\text{Area} = \left(\frac{1}{5}2^{5/2} - \frac{2}{3}2^{3/2}\right) - (0)$$

$$\text{Area} = \boxed{\frac{1}{5}(\sqrt{2})^5 - \frac{2}{3}(\sqrt{2})^3}$$

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vs "old" way

$\boxed{x = t^2 - 2t,\ y = \sqrt{t}}$ and (y-axis)

$$\int_c^d x(y)\,dy$$

$\to x = t^2 - 2t,\quad y^2 = t$

$$x = y^4 - 2y^2$$

$$\text{Area} = \int_0^{\sqrt{2}} (y^4 - 2y^2)\,dy = \frac{1}{5}y^5 - \frac{2}{3}y^3 \Big|_0^{\sqrt{2}}$$

$$= \left[\frac{1}{5}(\sqrt{2})^5 - \frac{2}{3}(\sqrt{2})^3\right]$$

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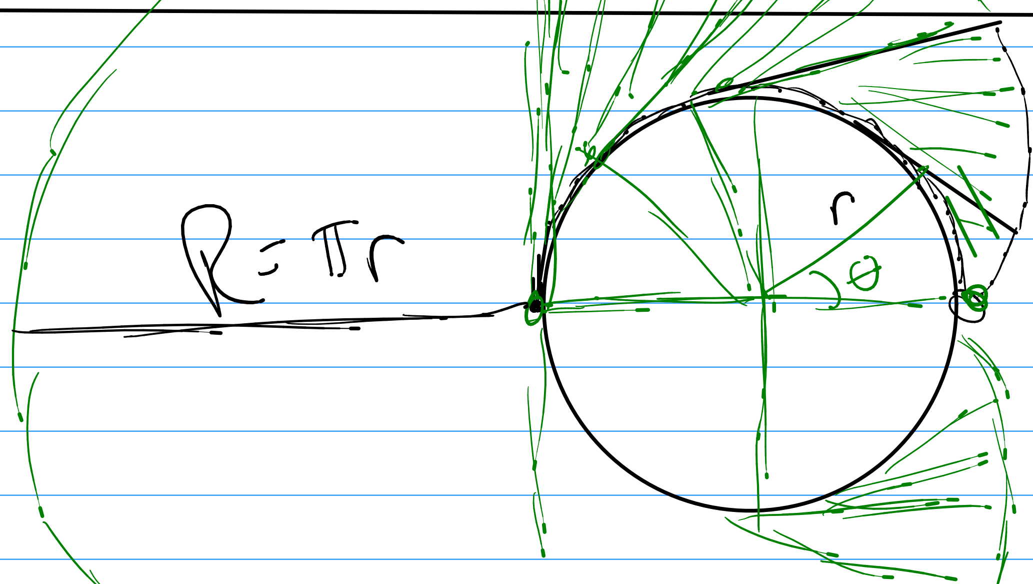
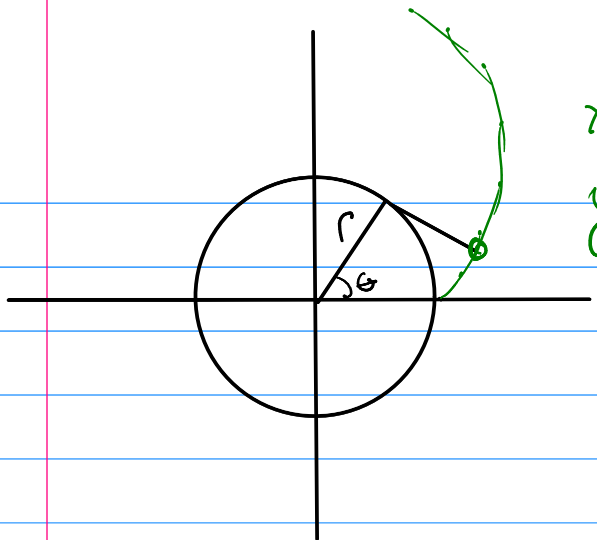
ex #74 p.697

#73

$\to$ involute





$x = r(\cos\theta + \theta\sin\theta)$

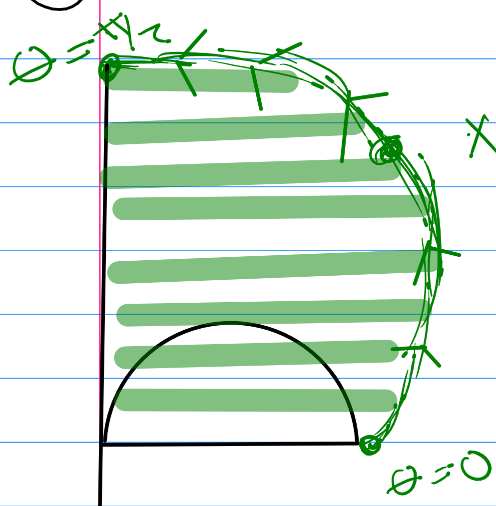
$y = r(\sin\theta - \theta\cos\theta)$

Area = 2(Area of involute from $\theta = 0$ to $\theta = \pi$)

+ Area of half circle

① Area of ½ circle $= \pi(R) = \pi(\pi r) = r\pi^2$

② Involute − ½ silo area



$x = r(\cos\theta + \theta\sin\theta)$

$y = r(\sin\theta - \theta\cos\theta)$

$$\int_\alpha^\beta x\,dy = \int_\alpha^\beta x(\theta)\,y'(\theta)\,d\theta$$

$$y' = r(\cancel{\cos\theta} - \cancel{\cos\theta} + \theta\sin\theta])\,d\theta$$

$$dy = r\,\theta\sin\theta\,d\theta$$

$$r^2\int_0^{\pi/2}(\cos\theta + \theta\sin\theta)\,\theta\sin\theta\,d\theta$$