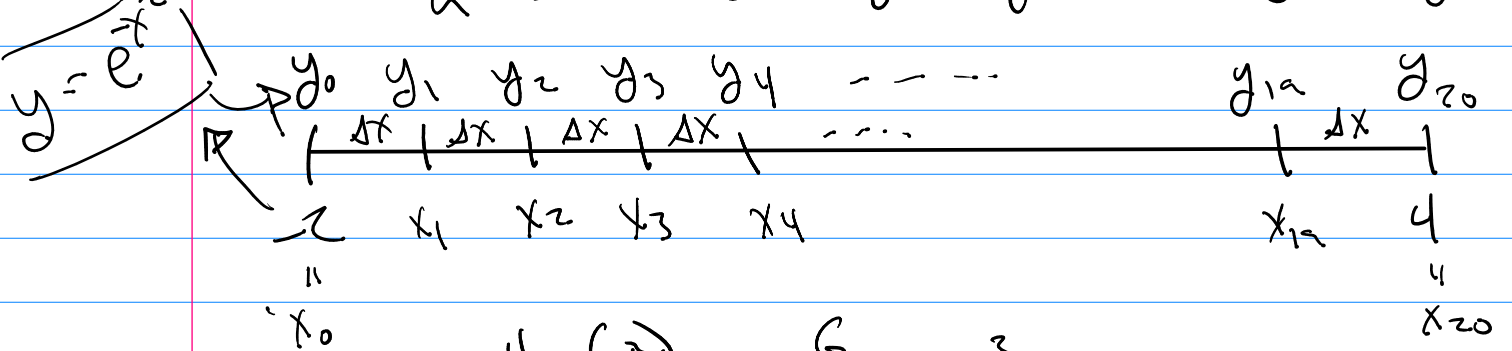
Math 243

Q's/ Approx Ints $\int_{-2}^{4} e^{-x^2}\,dx$ let $n = 20$

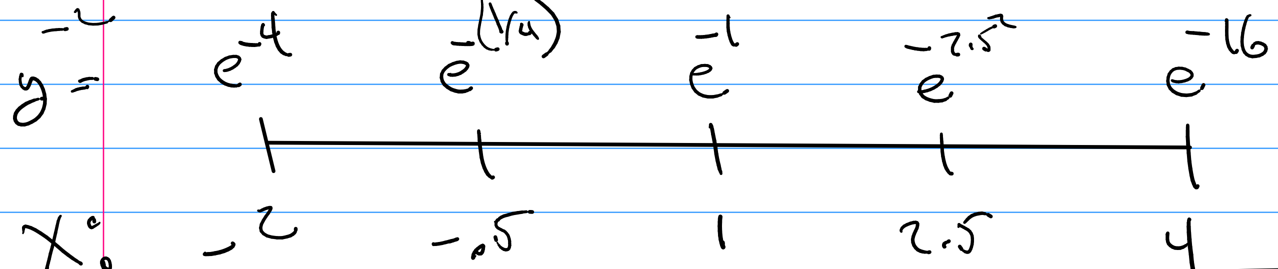
$$S_{20} = \frac{\Delta x}{n}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 4y_{19} + y_{20}\right)$$



$$\Delta x = \frac{4-(-2)}{20} = \frac{6}{20} = \frac{3}{10} = 0.3$$

See video

$\int_{-2}^{4} e^{-x^2}\,dx$ $n = 4$ $\Delta x = \frac{6}{4} = 1.5$



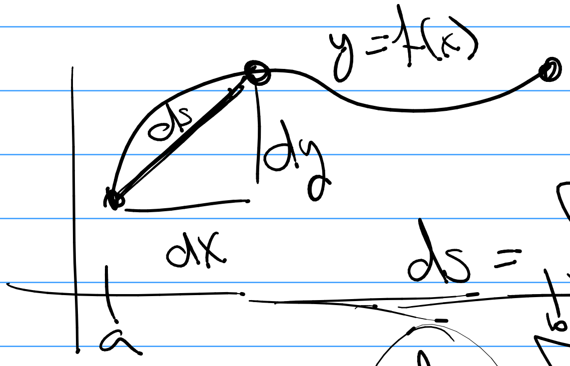
$$\boxed{S_4 = \frac{1.5}{3}\left(e^{-4} + 4e^{-1/4} + 2e^{-1} + 4e^{-2.5^2} + e^{-16}\right)}$$

$$T_4 = \frac{1.5}{2}\left(e^{-4} + 2e^{-1/2} + 2e^{-1} + 2e^{-2.5^2} + e^{-16}\right)$$

$$L_4 = 1.5\left(e^{-4} + e^{-1/2} + e^{-1} + e^{-2.5^2}\right)$$

Ch 8 Applications of Integration

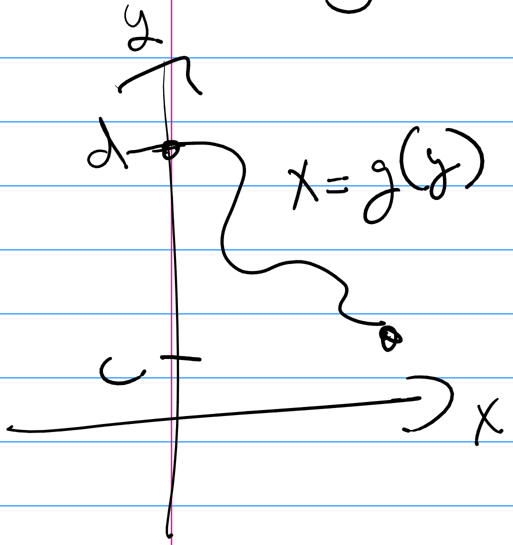
8.1 Arc length



$$ds = \sqrt{dx^2 + dy^2}$$

$$\boxed{ds} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$S = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

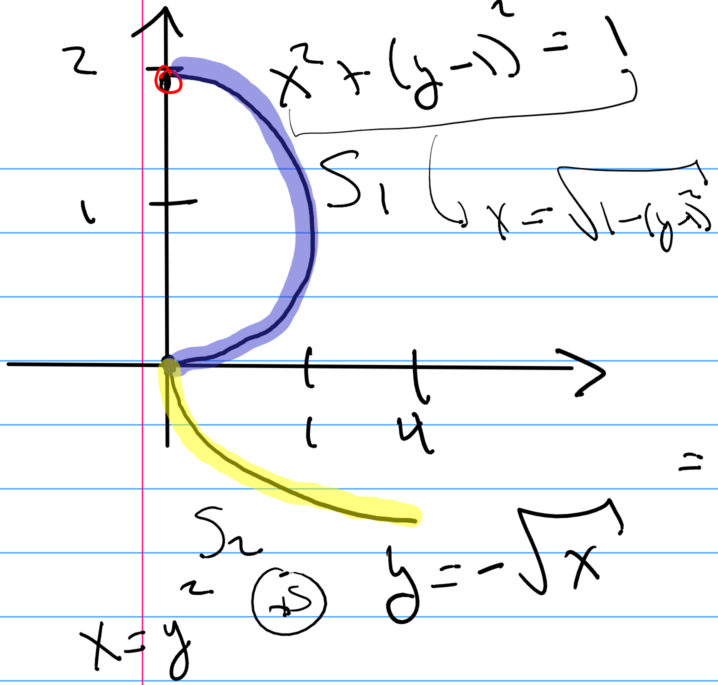


$$S = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$S = \int_c^d \sqrt{1+(g'(y))^2}\,dy$$

$$S = \int_{x=a}^{x=b} ds = \int_{x=a}^{x=b} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$S = \int_{y=c}^{y=d} ds = \int_{y=c}^{y=d} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$



$$S_1 = \int_{y=0}^{y=2} ds = \int_{y=0}^{y=2} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$\frac{dx}{dy} = \frac{d}{dy}\left[\left(1-(y-1)^2\right)^{1/2}\right]$$

$$= \frac{1}{\cancel{2}}\left(1-(y-1)^2\right)^{-1/2}\left(-\cancel{2}(y-1)\right)$$

$$= \frac{1-y}{\sqrt{1-(y-1)^2}}$$

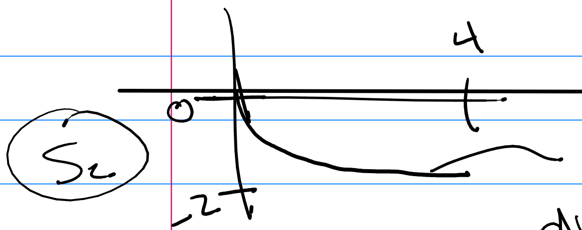
$$S_1 = \int_0^2 \sqrt{1+\frac{(1-y)^2}{1-(y-1)^2}}\, dy = \int_0^2 \sqrt{\frac{1}{1-(y-1)^2}}\, dy$$

table → #16

$$S_1 = \int_0^2 \frac{1}{\sqrt{1-(y-1)^2}}\, dy = \int_{-1}^{1} \frac{1}{\sqrt{1-u^2}}\, du$$

$$S_1 = \arcsin(u)\Big|_{-1}^{1} = \arcsin(1) - \arcsin(-1)$$

$$= \left(\pi/2\right) - \left(-\pi/2\right) = \boxed{\pi}$$



$y=-\sqrt{x}$ or $x=y^2$

$$\frac{dy}{dx} = \frac{-1}{2\sqrt{x}} \qquad \frac{dx}{dy} = 2y$$

$$\int_0^4 \sqrt{1+\left(-\frac{1}{2\sqrt{x}}\right)^2}\, dx = \int_0^{-2} \sqrt{1+(2y)^2}\, dy$$

#1 #2

#1 $\int_0^4 \sqrt{\frac{4x+1}{4x}}\,dx = \frac{1}{2}\int_0^4 \boxed{\frac{\sqrt{4x+1}}{\sqrt{x}}}\,dx$

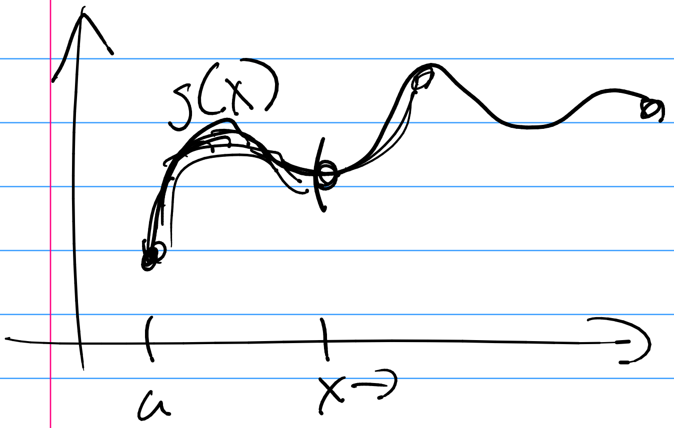
#2 $\int_0^{-2} \sqrt{1+(2y)^2}\,dy = 2\boxed{\int_0^{-2} \sqrt{1/4+y^2}\,dy}$

use table #21

$$\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right)+C$$

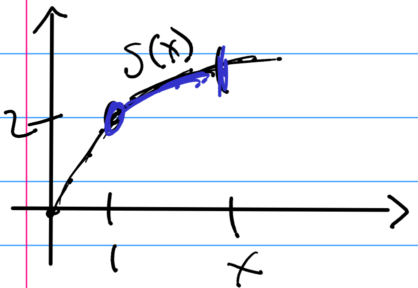
Arc length function

ex $d = r \cdot t \rightarrow r = \frac{d}{t}$



$$S(x) = \int_a^x \sqrt{1+\left(\frac{dy}{dt}\right)^2}\,dt$$

$f(\square) = 2\square^{3/2}$



$y = 2x^{3/2}$

$$S(x) = \int_1^x \sqrt{1+(f'(t))^2}\,dt$$

$f(t) = 2t^{3/2} \quad f'(t) = 3t^{1/2}$

$$S(x) = \int_{t=1}^{t=x} \sqrt{1+9t}\, dt = \frac{1}{9}\int_{10}^{1+9x} \sqrt{u}\, du$$

$$\boxed{\begin{aligned} \text{let } u &= 1+9t \\ du &= 9dt \end{aligned}}$$

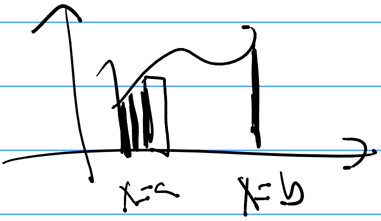
$$S(x) = \frac{1}{9}\left(\frac{2}{3}u^{3/2}\right)\Bigg|_{10}^{1+9x} = \frac{2}{27}\left[(1+9x)^{3/2} - 10^{3/2}\right]$$

$\boxed{8.2}$ Surface of revolution

Idea of slicing

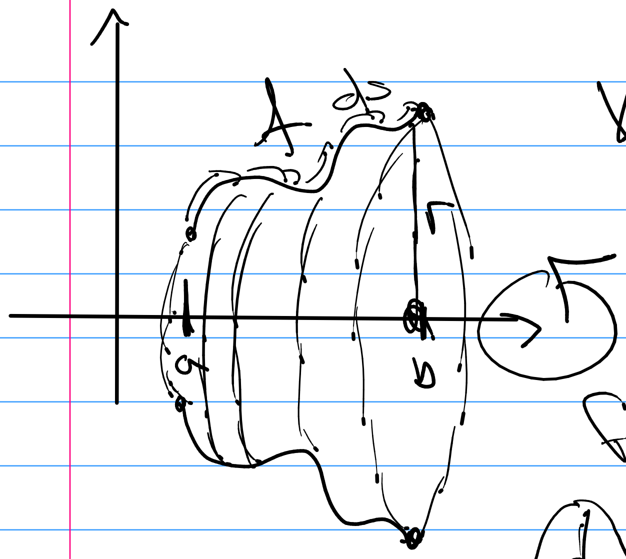
$$\int_{x=a}^{x=b} \square$$

$$\int_a^b \underline{f(x)}\,\boxed{dx}$$



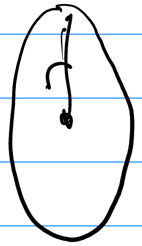
$$\boxed{\int_a^b}\ \boxed{f(x)\cdot dx}$$

add the rectangles



$$V = \int_a^b (\pi r^2)\,dx = \int_a^b \pi (f(x))^2\,dx$$

for surface area the slice is..

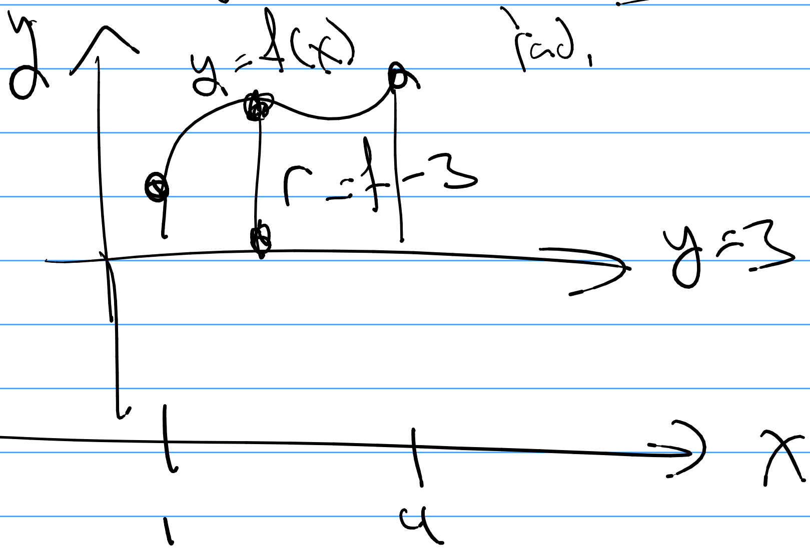


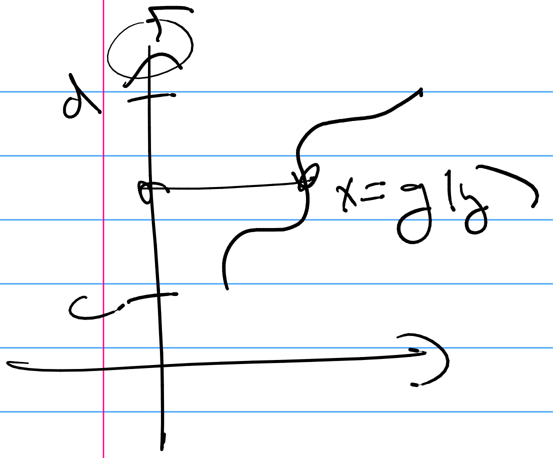
the circum. of the circle.

$$C = 2\pi r = 2\pi f(x)$$

$$SA = \int_a^b \underbrace{(2\pi r)}_{\text{circ's}}(\text{arc length}) = \int_{x=a}^{x=b} (\text{circum})(ds)$$

$$SA = \int_a^b 2\pi \underbrace{f(x)}_{\text{rad.}} \underbrace{\boxed{\sqrt{1+(f'(x))^2}\,dx}}_{\text{arc length}}$$





$$SA = \int_c^d 2\pi \underbrace{\boxed{g(y)}}_{\text{radius}} \underbrace{\sqrt{1+(g'(y))^2}\,dy}_{\text{arc length}}$$

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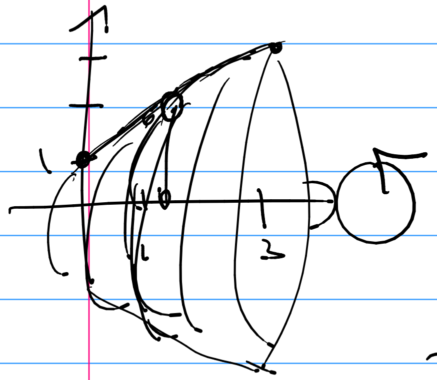
ex $y = \sqrt{x^2+1}$ $0 \le x \le 3$ about x-axis

$y' = \frac{1}{2}(x^2+1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}}$

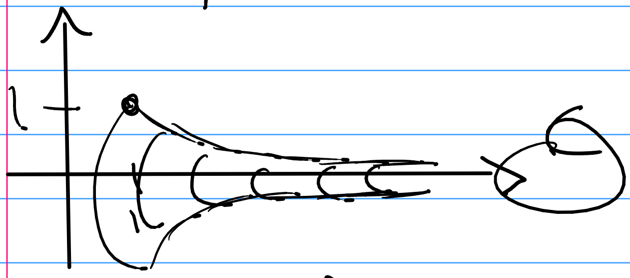


$$SA = \int_0^3 \underbrace{\left(2\pi\sqrt{x^2+1}\right)}_{\text{circum.}} \underbrace{\sqrt{1+\left(\frac{x}{\sqrt{x^2+1}}\right)^2}\,dx}_{\text{arc length}}$$

$$SA = 2\pi\int_0^3 \cancel{\sqrt{x^2+1}}\sqrt{\frac{2x^2+1}{\cancel{x^2+1}}}\,dx$$

$$= 2\pi\int_0^3 \sqrt{2x^2+1}\,dx \approx (7.299174527898268)2\pi$$

$f(x) = \frac{1}{x}$ from $x = 1$ to ∞



$$V = \int_1^\infty \pi r^2 \, dx = \pi \int_1^\infty \frac{1}{x^2} dx$$

$$V = \pi \lim_{b \to \infty} \int_1^b x^{-2} dx = \pi \lim_{b \to \infty} \left(-x^{-1}\right)\Big|_1^b$$

$$V = \pi \lim_{b \to \infty} \left(-\overset{0}{\cancel{\frac{1}{b}}} + 1\right) = \boxed{\pi} \text{ units}^3$$

$y = \frac{1}{x}$

$y' = -\frac{1}{x^2}$

VS

$$SA = \int_1^\infty 2\pi\left(\frac{1}{x}\right)\sqrt{1 + \left(-\frac{1}{x^2}\right)^2} \, dx$$

$$SA = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}} \, dx$$

$$SA = 2\pi \int_1^\infty \frac{\sqrt{x^4+1}}{x^3} dx = 2\pi \int_1^\infty x^{-3}\sqrt{x^4+1} \, dx$$

$$SA = 2\pi \lim_{b \to \infty} \int_1^b x^{-3}\sqrt{x^4+1} \, dx$$

$$\int \frac{\sqrt{x^4+1}}{x^3}\,dx = \int \frac{\sqrt{u^2+1}}{x^3}\,\frac{1}{2x}\,du$$

let $u = x^2$

$du = 2x\,dx$

$$= \frac{1}{2}\int \frac{\sqrt{u^2+1}}{u^2}\,du$$

$a = 1$

Forms of $\sqrt{a^2+u^2}$ $a>0$ #21 – #29

and #24 $\displaystyle\int \frac{\sqrt{a^2+u^2}}{u^2}\,du = -\frac{\sqrt{a^2+u^2}}{u} + \ln\left(u+\sqrt{a^2+u^2}\right) + C$

$$= \frac{1}{2}\left[-\frac{\sqrt{1+u^2}}{u} + \ln\left(u+\sqrt{1+u^2}\right)\right] + C$$

and $u = x^2$ so..

$$\int \frac{\sqrt{x^4+1}}{x^3}\,dx = -\frac{\sqrt{1+x^4}}{2x^2} + \frac{1}{2}\ln\left(x^2+\sqrt{1+x^4}\right) + C$$

Now:

$$SA = \pi \lim_{b\to\infty}\left[-\frac{\sqrt{1+x^4}}{x^2} + \ln\left(x^2+\sqrt{1+x^4}\right)\right]_1^b$$

$$SA = \pi \lim_{b\to\infty}\left[\left[-\overset{\to 1}{\cancel{\frac{\sqrt{1+b^4}}{b^2}}} + \overset{\to\infty}{\ln\left(\boxed{b^2} + \sqrt{1+b^4}\right)}\right] - \left[\underbrace{@1}_{\text{const}}\right]\right]$$

$$\boxed{SA \to \infty}$$

divergent

ex $\quad \int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\,dx \qquad \int_1^\infty g(x)\,dx = \boxed{\text{diverg}}$

Comparison

$$\frac{1}{x} = \frac{x^2}{x^3} = \frac{\sqrt{x^4}}{x^3} < \frac{\sqrt{x^4+1}}{x^3}$$

$$\int_1^\infty \frac{1}{x}\,dx \text{ is div} \to \int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\,dx \text{ is div}$$

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ex $\quad \int \frac{\sqrt{x^4+1}}{x}\,dx = \int \frac{u}{x}\,dx = \int \frac{u}{x}\,\frac{u\,du}{2x^3}$

let $u = \sqrt{x^4+1} \to x^4 = u^2 - 1$

$$du = \frac{1}{2}(x^4+1)^{-1/2}(\overset{2}{\cancel{4}}x^3)\,dx \to dx = \frac{\overset{=u}{(x^4+1)^{1/2}}}{2x^3}\,du$$

$$\to dx = \frac{u\,du}{2x^3}$$

$$\to = \frac{1}{2}\int \frac{u^2\,du}{x^4} = \frac{1}{2}\int \frac{u^2}{u^2-1}\,du$$

So $\frac{u^2}{u^2-1}$

$\to 1 + \frac{1}{u^2-1}$

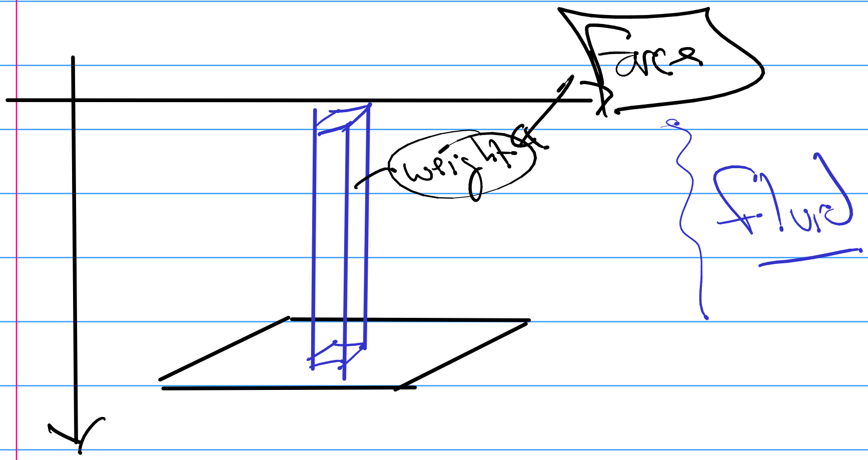
$$\begin{array}{r} 1 \\ u^2+0u-1 \overline{\smash{)}\, u^2+0u+0} \\ -(u^2+0u-1) \\ \hline 1 \end{array}$$

$$\frac{1}{2}\int\left(1+\frac{1}{u^2-1}\right)du = \frac{1}{2}u+\frac{1}{4}\ln\left|\frac{u-1}{u+1}\right|+C$$

$\boxed{bu}$ $u = \sqrt{x^4+1}$

$$= \frac{1}{2}\sqrt{x^4+1}+\frac{1}{4}\ln\left|\frac{\sqrt{x^4+1}-1}{\sqrt{x^4+1}+1}\right|+C$$

$\boxed{8.3}$ Applications to Physics and Engineering



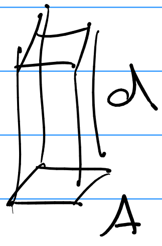
$$\text{Pressure} = \frac{\text{Force}}{\text{Area}}$$

Weight = Force of gravity upon my Mass

$$F = mg = \boxed{\rho A d}\quad g$$

$\rho = \frac{\text{mass}}{\text{units}}$

Mass density.



$V = Ad$

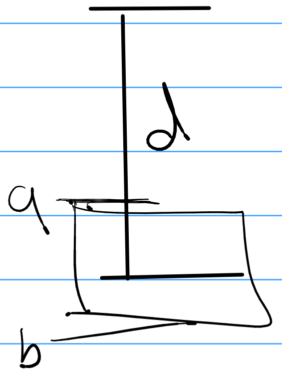
fluid has $\rho = \frac{\text{mass}}{\text{unit}^3}$ mass density

$$\boxed{F = \rho A d \cdot g}$$

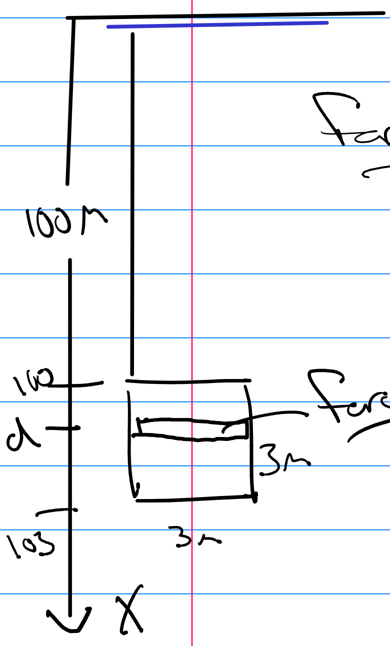
(force due to gravity)
"weight"

Pressure $= \frac{F}{A}$

$$\boxed{P = \rho d g}$$



$\Rightarrow P \cdot A = F$

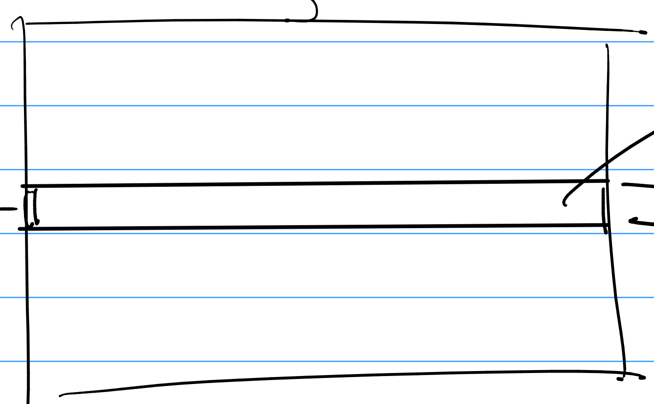


force?

Force on the plate?

$F = \underset{\substack{\uparrow \\ \rho d g}}{P} \cdot A$

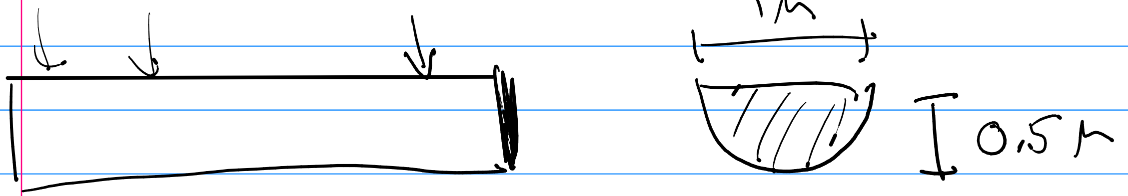
@ a fixed depth we have same pressure

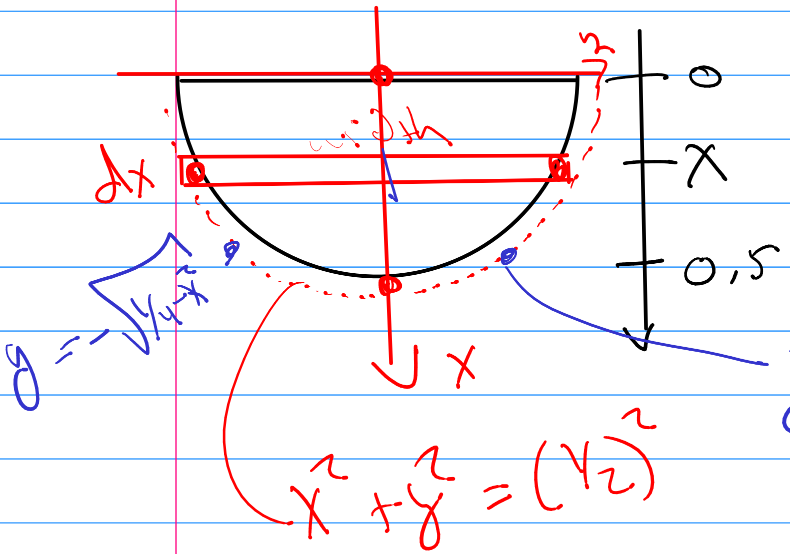


$F_i = P \cdot A$
$= \rho d g (3\,dx)$

$$\int_{100}^{103} (\rho x g)(3\,dx) = 3\rho g \int_{100}^{103} x\,dx$$

$$= 3\rho g\, \frac{1}{2}x^2 \Big|_{100}^{103} = \boxed{\frac{3}{2}\rho g\left(103^2 - 100^2\right)}$$





$$\int_0^{0.5} (\rho x g)(\text{area})$$

width dx

$y = \sqrt{1/4 - x^2}$ $\quad (2\sqrt{1/4 - x^2})\,dx$

$$F = \int_0^{0.5} \rho x g\, 2\sqrt{1/4 - x^2}\, dx$$

$$= 2\boxed{\rho} g \int_0^{1/2} x\sqrt{1/4 - x^2}\, dx = \text{Ayuh}$$

let $u = 1/4 - x^2$

$du = -2x\,dx$