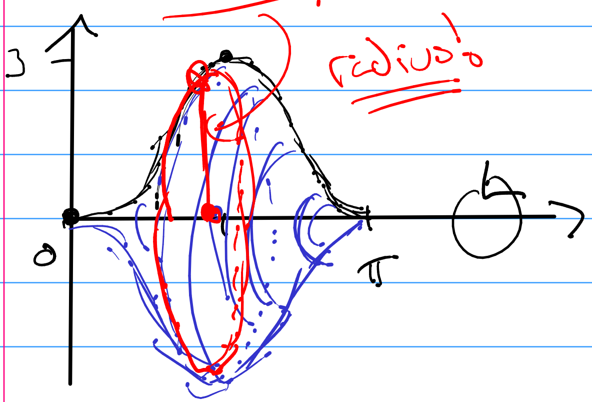
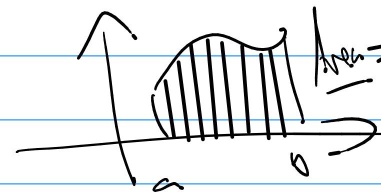
Math 243

Q3) y = 3 (sin(x))^2, y = 0, 0 ≤ x ≤ π;   about the x−axis



Note:



$$\text{Area} = \int_a^b [f(x)]\,dx$$

$$\text{Vol} = \int_0^{\pi} (\text{2D slices})\,dx$$

area of circle = $\pi(\text{rad})^2$

$$V = \int_0^{\pi} \pi\left(3\sin^2 x\right)^2 dx = 9\pi \int_0^{\pi} \sin^4 x\,dx$$

$$V = 9\pi \int_0^{\pi} \sin^4 x\,dx$$

$$(\sin^2 x)^2 = \left(\tfrac{1}{2}(1-\cos 2x)\right)^2 = \tfrac{1}{4}\left(1 - 2\cos 2x + \boxed{\cos^2 2x}\right)$$

$$\sin^4 x = \tfrac{1}{4} - \tfrac{1}{2}\cos 2x + \left(\tfrac{1}{2}(1+\cos 4x)\right)$$

$$= \boxed{\tfrac{3}{4} - \tfrac{1}{2}\cos 2x + \tfrac{1}{2}\cos 4x}$$

$$V = 9\pi \int_0^{\pi} \left(\boxed{\tfrac{3}{4}} - \boxed{\tfrac{1}{2}\cos 2x} + \boxed{\tfrac{1}{2}\cos 4x}\right) dx$$

= finish!

$$\int \frac{dx}{\sqrt{x} - 5\sqrt[3]{x}} \qquad \text{let } u = \sqrt[6]{x}$$

$$\int \frac{dx}{x^{1/2} - 5x^{1/3}} = \int \frac{dx}{x^{3/6} - 5x^{2/6}} = \int \frac{6u^5}{u^3 - 5u^2}\,du$$

let $u = x^{1/6}$

$du = \frac{1}{6}x^{-5/6}\,dx \rightarrow 6x^{5/6}\,du = dx$

$6u^5\,du = dx$

$$6\int \frac{u^5}{u^3 - 5u^2}\,du = 6\int \frac{u^3}{u-5}\,du = 6\int \left[u^2 + 5u + 25 + \frac{125}{u-5}\right]du$$

$$\begin{array}{r} u^2 + 5u + 25 \\ u-5\,\overline{)\,u^3 + 0u^2 + 0u + 0} \\ \underline{u^3 - 5u^2} \\ 5u^2 + 0u \\ \underline{5u^2 - 25u} \\ 25u + 0 \\ \underline{25u - 125} \\ 125 \end{array}$$

$$= 6\left[\frac{1}{3}u^3 + \frac{5}{2}u^2 + 25u + 125\ln|u-5|\right] + C$$

$$= 2u^3 + 15u^2 + 150u + 750\ln|u-5| + C$$

Remember $u = x^{1/6}$

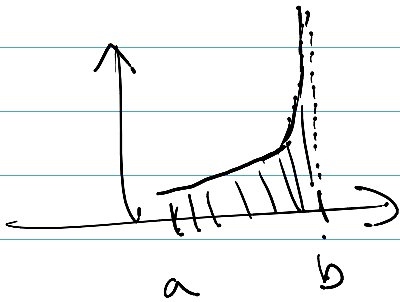
$$= \boxed{2\sqrt{x} + 15\sqrt[3]{x} + 150\sqrt[6]{x} + 750\ln\left|\sqrt[6]{x} - 5\right| + C}$$

7.8 Improper Definite Integrals.. $\int_a^b f(x)\,dx$

① 

$$\int_a^\infty f(x)\,dx$$

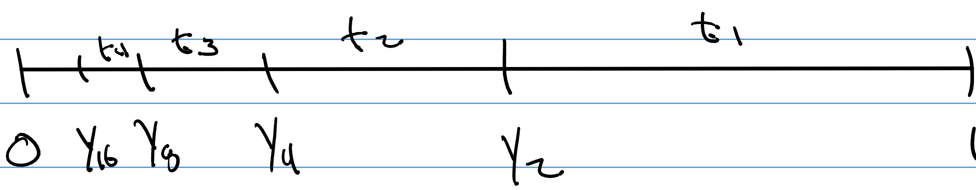
Area when width is not finite!

② 

$f(x)$ is discontinuous @ $x = b$

Area when height is not finite!

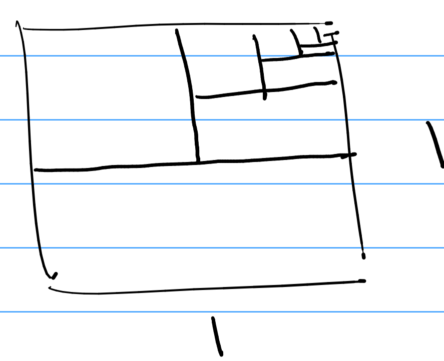
Zeno's Paradox



time for arrow to hit $t_1 + t_2 + t_3 + t_4 + \dots$

$t \to \infty$?

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = 1$$

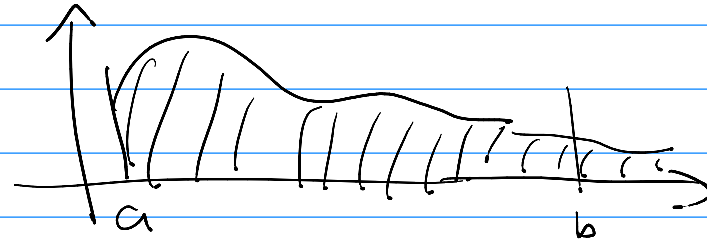


$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{2^i}$$

(1) Infinite Bounds $\int_a^b f(x)\,dx$

a and/or b is infinite

a) $\int_a^\infty f(x)\,dx$



if b was "large" $\int_a^b f(x)\,dx \approx \int_a^\infty f(x)\,dx$

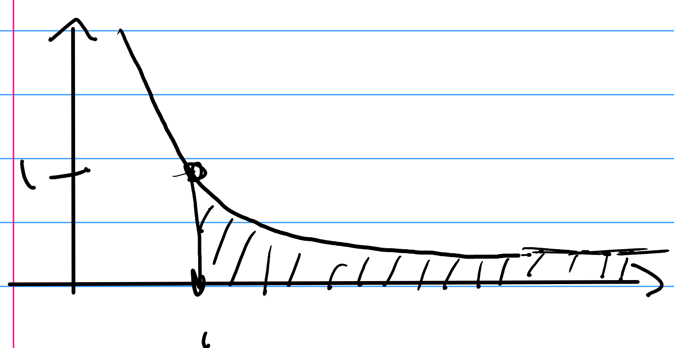
def $\boxed{\int_a^\infty f(x)\,dx} = \lim_{b\to\infty} \int_a^b f(x)\,dx$

→ limit exists, $\boxed{\text{converges}}$

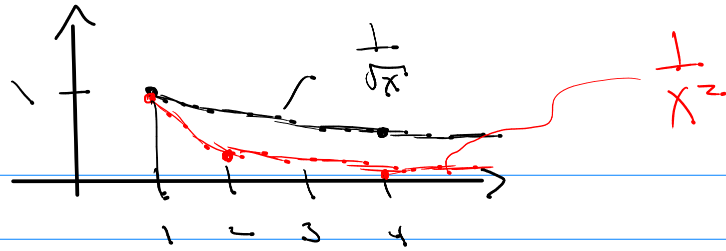
→ limit does not exist, $\boxed{\text{diverges}}$

$\boxed{\text{ex}}$ $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b x^{-2}\,dx$

$= \lim_{b\to\infty} \left(-x^{-1}\right)\Big|_1^b = \lim_{b\to\infty} \left(-\frac{1}{b}^{\to 0} + 1\right) = 1$

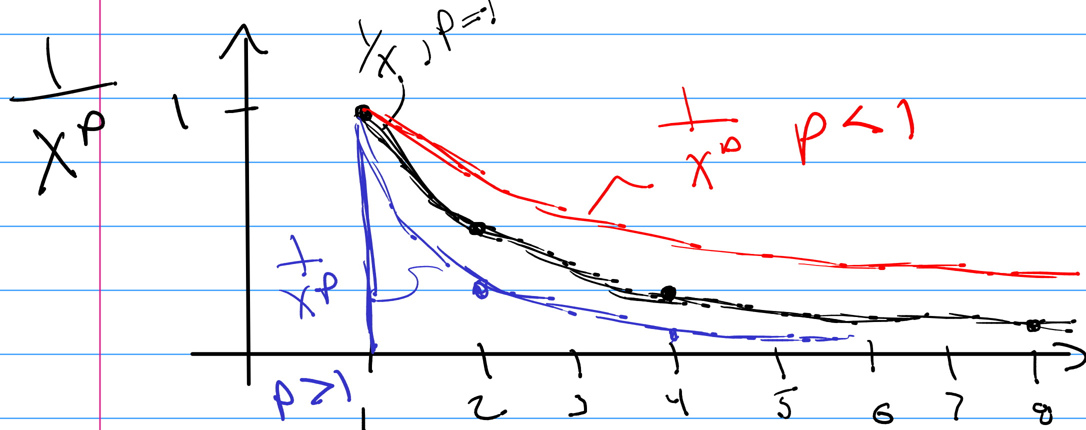


$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx$$



$$= \lim_{b\to\infty}\int_1^b x^{-1/2}\,dx = \lim_{b\to\infty}\left(2x^{1/2}\right)\Big|_1^b$$

$$= \lim_{b\to\infty}\left(2\sqrt{b}^{\to\infty} - 2\right) = \infty \quad \boxed{\text{diverges}}$$



$$\int_1^\infty \frac{1}{x^p}\,dx$$

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b x^{-p}\,dx$$

$$= \lim_{b\to\infty}\left(\frac{1}{1-p}x^{1-p}\right)\Big|_1^b = \lim_{b\to\infty}\left(\frac{1}{1-p}b^{1-p} - \frac{1}{1-p}\right)$$

$$= \frac{1}{1-p}\lim_{b\to\infty}\left(b^{1-p} - 1\right) = \begin{cases} p>1 \to \frac{1}{p-1} \\ p<1 \to \boxed{\text{div}} \end{cases}$$

$p=1$?

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \ln|x|\Big|_1^b = \lim_{b\to\infty}(\ln b) = \infty \quad \boxed{\text{div}}$$

$\boxed{S_0}$ $\int_1^{\infty} \frac{1}{x^p}\,dx$ diverges for $p \le 1$
converges for $p > 1$

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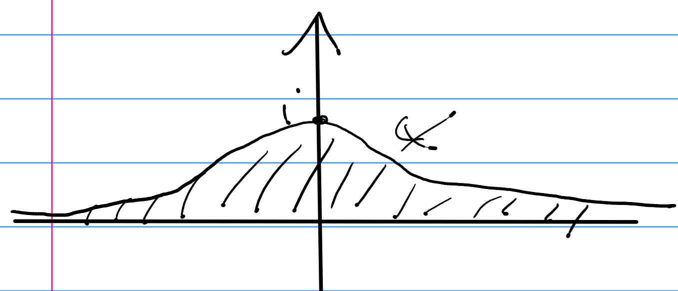
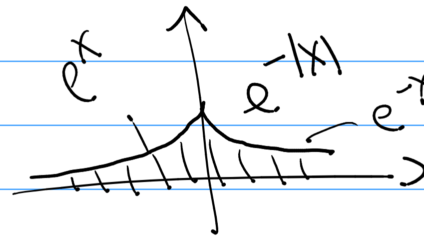
① $\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$

② $\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$

③ $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$





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ex. $\int_{-\infty}^{\infty} e^{-|x|}\,dx = \int_{-\infty}^{0} e^{-|x|}\,dx + \int_0^{\infty} e^{-|x|}\,dx$

Note: $|x| = \begin{cases} x & x \ge 0 \\ -x & x < 0 \end{cases}$

$= \int_{-\infty}^{0} e^{-(-x)}\,dx + \int_0^{\infty} e^{-(x)}\,dx$

$$= \int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{\infty} e^{-x}\,dx$$

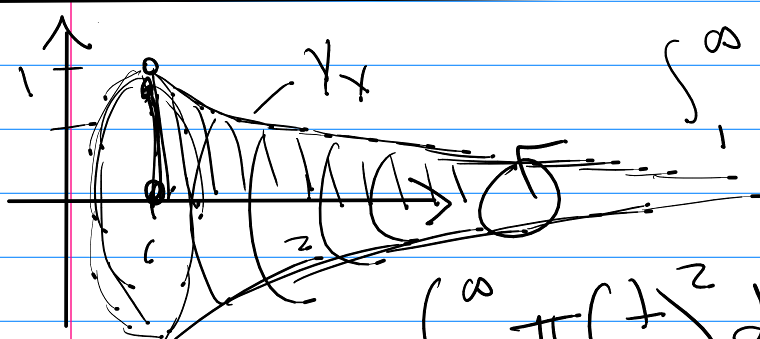
$$= \lim_{a\to-\infty} \int_{a}^{0} e^{x}\,dx + \lim_{b\to\infty} \int_{0}^{b} e^{-x}\,dx$$

$$= \lim_{a\to-\infty} \left[e^{x} \Big|_{a}^{0} \right] + \lim_{b\to\infty} \left[-e^{-x} \Big|_{0}^{b} \right]$$

$$= \lim_{a\to-\infty} \left[1 - \cancel{e^{a}}^{\,0} \right] + \lim_{b\to\infty} \left[-\cancel{e^{-b}}^{\,0} + 1 \right]$$

$$= 1 + 1$$

$$= \boxed{2}$$



$$\int_{1}^{\infty} \frac{1}{x}\,dx$$ divergent.

$$\int_{1}^{\infty} \pi \left(\frac{1}{x}\right)^{2} dx = \pi \int_{1}^{\infty} \frac{1}{x^{2}}\,dx$$

$$\pi \lim_{b\to\infty} \int_{1}^{b} x^{-2}\,dx = \pi \lim_{b\to\infty} -x^{-1} \Big|_{1}^{b}$$

$$= \pi \lim_{b\to\infty} \left[-\cancel{\frac{1}{b}}^{\,0} + 1 \right] = \boxed{\pi}$$

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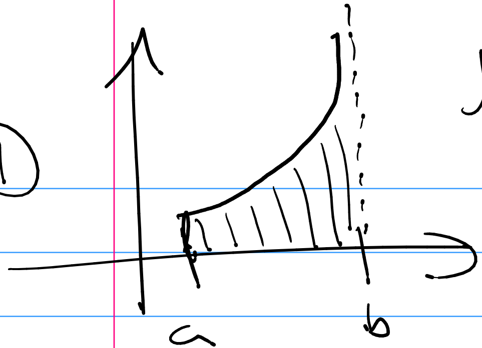
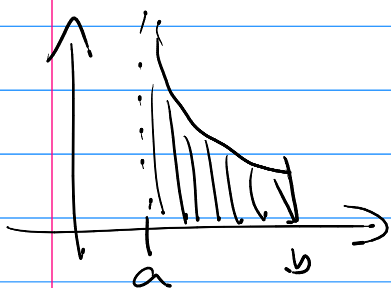
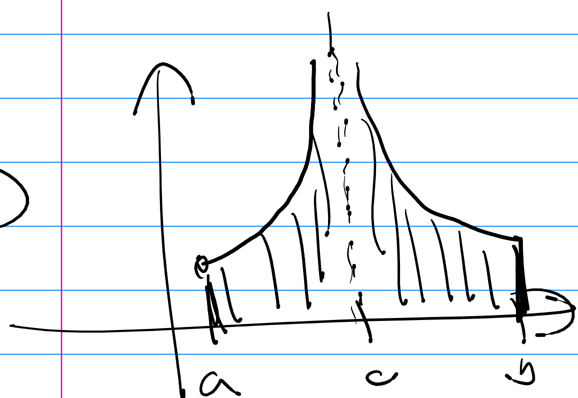
1 

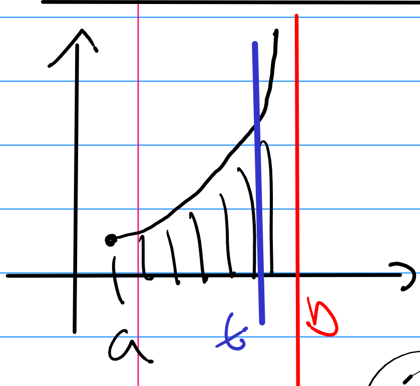
$f(x)$ is discontinuous @ $x=b$

2 

$f(x)$ is disc. @ $x=a$

3 

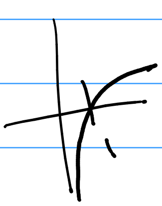
$f(x)$ is disc. @ $x=c$ in the $[a,b]$ interval



$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

2x

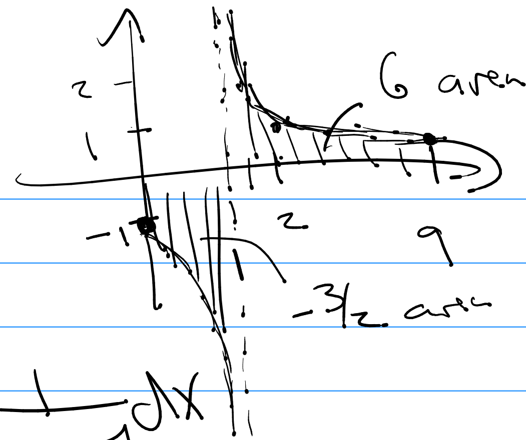
$$\int_0^2 \frac{1}{x-2}\,dx = \lim_{t\to 2^-}\boxed{\int_0^t \frac{1}{x-2}\,dx}$$



$$= \lim_{t\to 2^-} \ln|x-2|\Big|_0^t = \lim_{t\to 2^-} \ln|t-2| - \ln 2$$

diverges

ex $\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$



$$= \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^9 \frac{1}{\sqrt[3]{x-1}}\,dx$$

$$= \lim_{s\to 1^-} \int_0^s \frac{1}{\sqrt[3]{x-1}}\,dx + \lim_{t\to 1^+} \int_t^9 \frac{1}{\sqrt[3]{x-1}}\,dx$$

both use

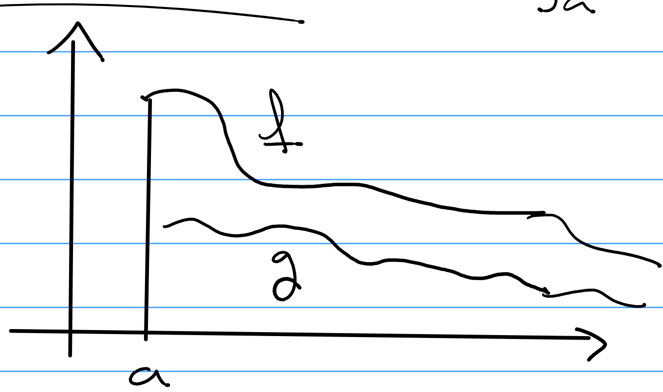
$$\boxed{\begin{aligned} \int (x-1)^{-1/3}\,dx &= \int u^{-1/3}\,du \\ \text{let } u = x-1 \quad &= \tfrac{3}{2}u^{2/3} + C \\ du = dx \quad &= \tfrac{3}{2}(x-1)^{2/3} + C \end{aligned}}$$

$$= \lim_{s\to 1^-}\left[\tfrac{3}{2}\cancel{(s-1)^{2/3}}^{\,0} - \tfrac{3}{2}\right] + \lim_{t\to 1^+}\left[\tfrac{3}{2}(4) - \tfrac{3}{2}\cancel{(t-1)^{2/3}}^{\,0}\right]$$

$$= -\tfrac{3}{2} \qquad + \qquad 6$$

$$= \boxed{9/2}$$

Comparison theorem $\displaystyle\int_a^\infty f(x)\,dx$ vs $\displaystyle\int_a^\infty g(x)\,dx$



$f(x) \geq g(x)$

on $x \geq a$

Thm ① if area under f is conv.

$\Rightarrow$ g is also convergent.

② if area under g is div.

$\Rightarrow$ f is also divergent.

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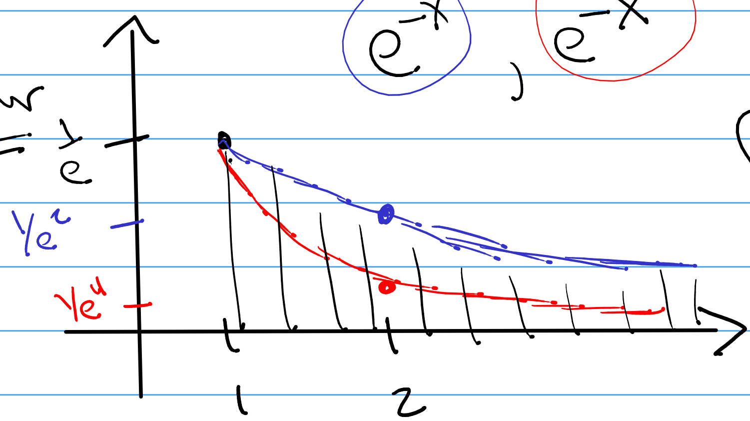
ex $\displaystyle\int_1^\infty e^{-x^2}\,dx = \lim_{b\to\infty} \boxed{\int_1^b e^{-x^2}\,dx}$

Can't do this!

Consider e^{-x}, e^{-x^2}



for $\boxed{x \geq 1}$

$x \leq x^2$

$-x \geq -x^2$

$e^{-x} \geq e^{-x^2}$

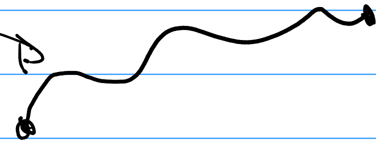
So $\displaystyle\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_1^b e^{-x}\,dx$

$\displaystyle = \lim_{b\to\infty} \left[-e^{-x} \Big|_1^b \right] = \lim_{b\to\infty} \left[\cancel{-e^{-b}}^{\,0} + 1/e \right] = 1/e$ $\boxed{\text{converges}}$

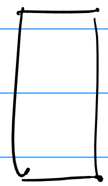
$\rightarrow$ $\boxed{\displaystyle\int_1^\infty e^{-x^2}\,dx \text{ also converges}}$

Ch 8 More Applications

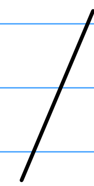
8.1 Arc length

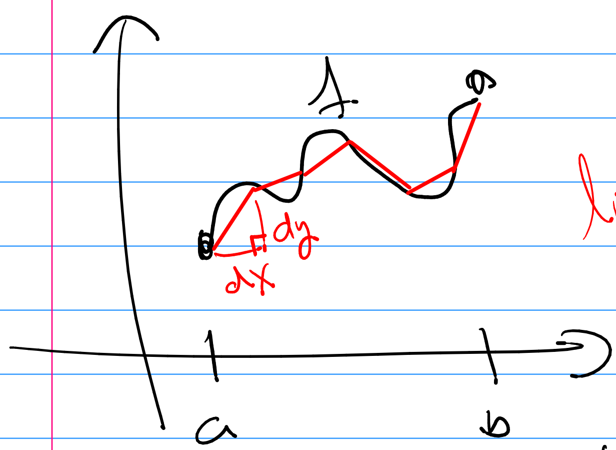


area



length





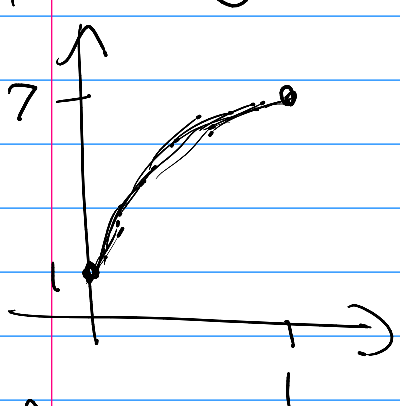
Sum of the lines ≈ arc length

$$\text{line } 1 = \sqrt{dx^2 + dy^2}$$

$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\text{arc length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

ex. $y = 1 + 6x^{3/2}$ from $x=0$ to $x=1$



$$y' = 9x^{1/2}$$

$$AL = \int_0^1 \sqrt{1 + (9\sqrt{x})^2}\, dx$$

$$= \int_0^1 \sqrt{1 + 81x}\, dx$$

let $u = 1 + 81x$

$du = 81\,dx$

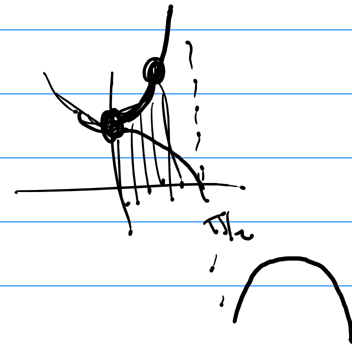
$$= \frac{1}{81}\int_1^{82} \sqrt{u}\, du = \frac{2}{243} u^{3/2}\Big|_1^{82}$$

$$= \frac{2}{243}\left[82^{3/2} - 1\right]$$

ex $y = \ln(\cos x)$ from $x=0$ to $x=\pi/3$

$$AL = \int_a^b \sqrt{1+(y')^2}\,dx$$

$$\to y' = \frac{1}{\cos x}(-\sin x) = -\tan x$$



$$\to AL = \int_0^{\pi/3} \sqrt{1+\tan^2 x}\,dx$$

$$= \int_0^{\pi/3} \sqrt{\sec^2 x}\,dx = \int_0^{\pi/3} |\sec x|\,dx$$

$$= \int_0^{\pi/3} \sec x\,dx = \ln|\sec x + \tan x|\Big|_0^{\pi/3}$$

$$= \boxed{\text{do it}}$$

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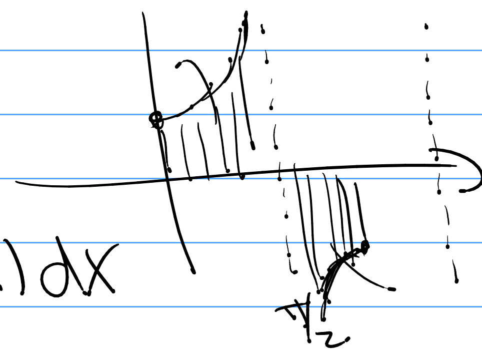
vs $\displaystyle\int_0^{\pi} |\sec x|\,dx$



$$= \int_0^{\pi/2} |\sec x|\,dx + \int_{\pi/2}^{\pi} |\sec x|\,dx$$

$$= \int_0^{\pi/2} \sec x\,dx + \int_{\pi/2}^{\pi} -\sec x\,dx$$

$$= \lim_{s\to\pi/2^-}\left[\int_0^{s} \sec x\,dx\right] + \lim_{t\to\pi/2^+}\left[\int_t^{\pi} -\sec x\,dx\right]$$

$=$ finish
