Math 243

Q5/ by parts:

$$\int (f)(g')\,dx = \boxed{fg} - \boxed{\int g f'\,dx}$$

$$f = \boxed{} \xrightarrow{\text{deriv}} f' = \boxed{}$$

$$g' = \boxed{} \xrightarrow{\text{Int}} g = \boxed{}$$

$$\int (\ )(\ )\,dx = fg - \boxed{\int \square\,\square\,dx}$$

$$= fg - \boxed{}$$

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ex

$$\int (\ln(x))^2\,dx = \boxed{\int u^2 e^u\,du} =$$

by parts to continue

sub?

let $u = \ln x \rightarrow x = e^u$

$du = \frac{1}{x}dx \rightarrow e^u du = dx$

or

$$\int \ln(x)\ln(x)\,dx = x(\ln x)^2 - x(\ln x) - \int(\ln x - 1)\,dx$$

by parts?

$f = \ln x \xrightarrow{\text{deriv}} f' = 1/x$

$g' = \ln x \xrightarrow{\text{int}} g = x\ln x - x$

$$\int \ln x\,dx = x\ln x - x + c$$

$f = \ln x \rightarrow f' = \frac{1}{x}$

$g' = 1 \qquad g = x$

$$\int (\ln x)^2\,dx = x(\ln x)^2 - x(\ln x) - [(x\ln x - x) - x] + C$$

$$= \boxed{x(\ln x)^2 - 2x\ln x + 2x + C}$$

Check: $\left[(\ln x)^2 + \cancel{2x\ln x\tfrac{1}{x}}\right] - \left[\cancel{2\ln x} + \cancel{2x\left(\tfrac{1}{x}\right)}\right] + [\cancel{2}] + 0$

$$= (\ln x)^2 \checkmark$$

$$\int u^2 e^u\,du = u^2 e^u - 2\int u e^u\,du$$

by parts

$f = u^2 \xrightarrow{\text{deriv}} f' = 2u$

$g' = e^u \xrightarrow{\text{Int}} g = e^u$

$$= u^2 e^u - 2\boxed{\int u e^u\,du}$$

$f = u \xrightarrow{\text{deriv}} f' = 1$

$g' = e^u \xrightarrow{\text{Int}} g = e^u$

$$= u^2 e^u - 2\left[u e^u - \int e^u\,du\right]$$

$$= u^2 e^u - 2u e^u + 2e^u + C$$

Note: From above prob $u = \ln x$

$$= x(\ln x)^2 - 2x\ln x + 2x + C$$

(e4) Partial fraction decomposition.

$$\int \frac{2x^3 - x^2 + 8x - 8}{x^4 - 2x^3 + 4x^2 - 8x}\,dx$$

(1) factor $x^4 - 2x^3 + 4x^2 - 8x = x(x^3 - 2x^2 + 4x - 8)$

$= x\left[x^2(x-2) + 4(x-2)\right]$

$= x(x-2)(x^2+4)$

(2) Decompose

$$\frac{2x^3 - x^2 + 8x - 8}{(x)(x-2)(x^2+4)} = \frac{c_1}{x} + \frac{c_2}{x-2} + \frac{c_3x + c_4}{x^2+4}$$

RHS. Numerator $= c_1(x-2)(x^2+4) + c_2x(x^2+4) + (c_3x+c_4)x(x-2)$

$= c_1(x^3 - 2x^2 + 4x - 8) + c_2(x^3 + 4x) + (c_3x + c_4)(x^2 - 2x)$

$= c_1x^3 - 2c_1x^2 + 4c_1x - 8c_1 + c_2x^3 + 4c_2x + c_3x^3 + (c_4 - 2c_3)x^2 - 2c_4x$

$= (c_1 + c_2 + c_3)x^3 + (c_4 - 2c_3 - 2c_1)x^2 + (4c_1 + 4c_2 - 2c_4)x - 8c_1$

LHS. Num $= (2)x^3 + (-1)x^2 + (8)x - 8$

const : $-8c_1 = -8 \rightarrow \boxed{c_1 = 1}$

x : $4c_1 + 4c_2 - 2c_4 = 8$

x^2 : $c_4 - 2c_3 - 2c_1 = -1$

x^3 : $c_1 + c_2 + c_3 = 2$

$\rightarrow$

$4\boxed{c_2} - 2c_4 = 4$

$c_4 - 2c_3 = 1$

$c_2 + c_3 = 1$

$c_2 = 1 - c_3$

$$\begin{cases} 4(1 - c_3) - 2c_4 = 4 \\ c_4 - 2c_3 = 1 \end{cases}$$

$$\left[\begin{array}{l} 2c_3 + c_4 = 0 \\ -2c_3 + c_4 = 1 \end{array}\right. \qquad 2c_3 = -1/2$$

$$2c_4 = 1 \qquad c_4 = 1/2$$

$c_3 = -1/4$

$c_2 = 5/4$

$c_1 = 1$

so

$$\int \left(\frac{1}{x} + \frac{5/4}{x-2} + \frac{\boxed{-1/4 x + 1/2}}{x^2+4} \right) dx$$

$$= \ln|x| + \frac{5}{4}\ln|x-2| + \int \frac{-1/4 x}{x^2+4}\,dx + \int \frac{1/2}{x^2+4}\,dx$$

$$= \ln|x| + \frac{5}{4}\ln|x-2| - \frac{1}{8}\int \frac{2x}{x^2+4}\,dx + \frac{1}{2}\boxed{\int \frac{1}{x^2+2^2}\,dx}$$

$$= \ln|x| + \frac{5}{4}\ln|x-2| - \frac{1}{8}\ln|x^2+4| + \boxed{\frac{1}{4}\tan^{-1}\left(\frac{x}{2}\right) + C}$$

ref. page 6 #17 $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ use here

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using "technology"

↳ or other peoples work, really.

① What? Maxima, geogebra, python (symbolic)
SAGE Math, Octave

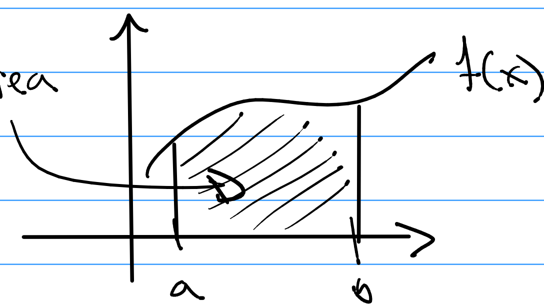
② when? you can do the problem... you don't want to.

(a) or you are learning about what you could try?

7.7 $\int e^{x^2}\,dx$ = non-elem function

$$\frac{d}{dx}\left[\;\square\;\right] = e^{x^2}$$

$\int_a^b f(x)\,dx$ = area



$$\int_a^b f(x)\,dx = F(b) - F(a)$$ where $\frac{d}{dx}\left[F(x)\right] = f(x)$

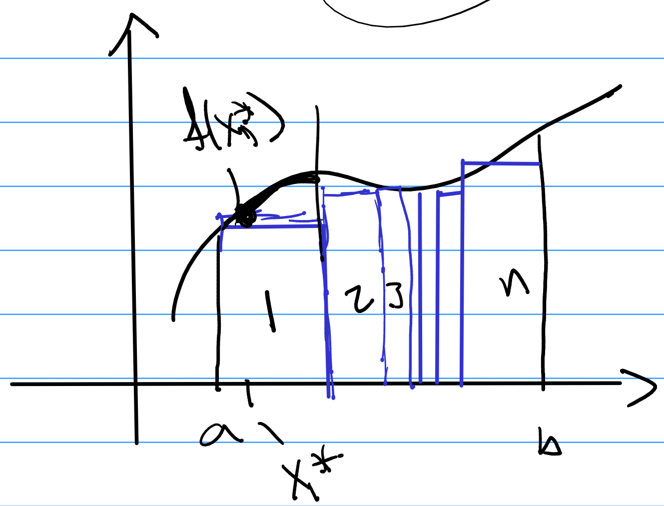
$$\int_a^x f(t)\,dt = F(x) - F(a)$$

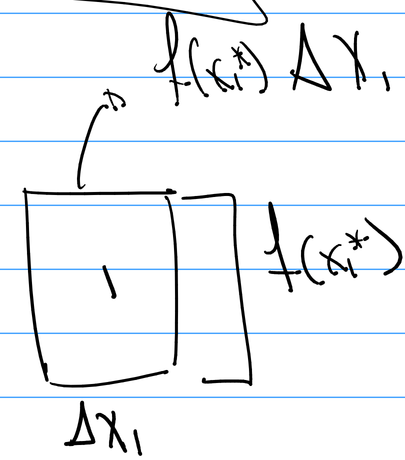
Say f(x)'s (one of) anti deriv. $F(x) = \boxed{\int_a^x f(t)\,dt}$

where $F(a) = 0$

$F(3) = \int_a^3 f(t)\,dt$ find area = find $F(3)$

b/c $\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$





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area $\approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$ for a fixed n.

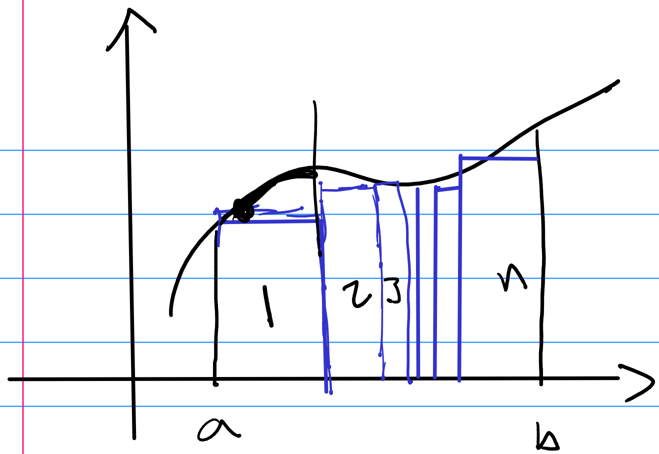
(big n $\to$ better answer)

$\Rightarrow F(x) = \int_a^x f(t)\,dt$

$F(x) \approx \sum_{i=1}^{n} f(t_i^*)\,\Delta t_i$ over $[a, x]$

So we need to be able to approx

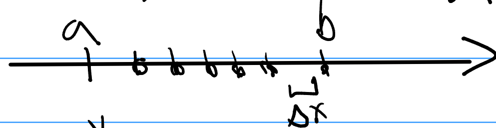
$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i \quad \text{over } [a,b]$$



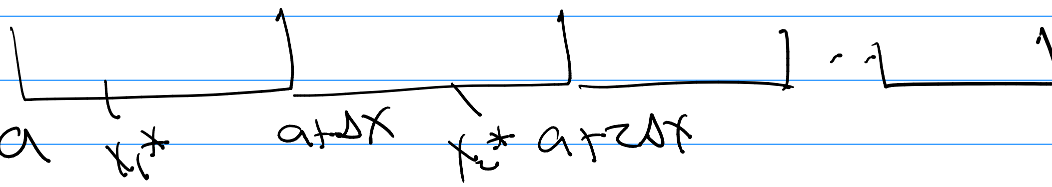
x_i^* ?

Δx_i ?

#1 $\Delta x_i = \Delta x = \frac{b-a}{n}$ (uniform partition (quadrature))



#2 pick x_i^*



a) pick left side (left end point approximation)

$x_1^* = a$

$x_2^* = a + \Delta x$

$x_3^* = a + 2\Delta x$

$\vdots$

$x_n^* = a + (n-1)\Delta x$

$$L_n = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

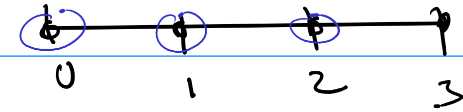
$$L_n = \Delta x \left[f(a) + f(a+\Delta x) + \cdots + f(a+(n-1)\Delta x) \right]$$

$$L_n = \Delta x \sum_{i=0}^{n-1} f(a + i\Delta x)$$

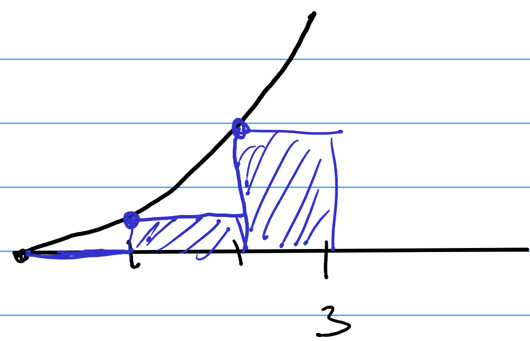
$$L_n = \frac{b-a}{n} \sum_{i=0}^{n-1} f(a + i\Delta x)$$

$$f(x) = \int_0^3 x^2\,dx = \frac{1}{3}x^3\Big|_0^3 = \frac{1}{3}3^3 = 9$$

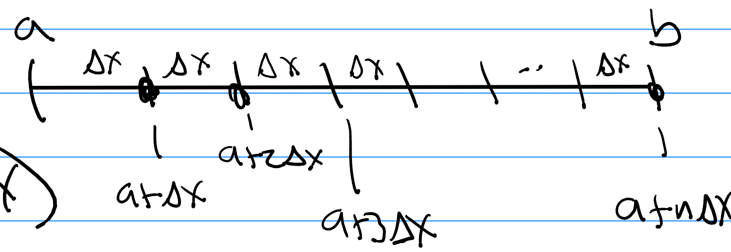
$n = 3 \qquad \Delta x = \frac{3-0}{3} = 3/3 = 1$



$$L_3 = (1)\left[0^2 + 1^2 + 2^2\right] = \boxed{5}$$

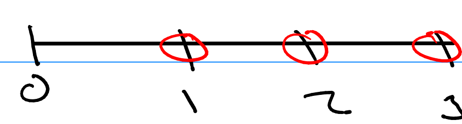


b) pick right side
(right end point approx)

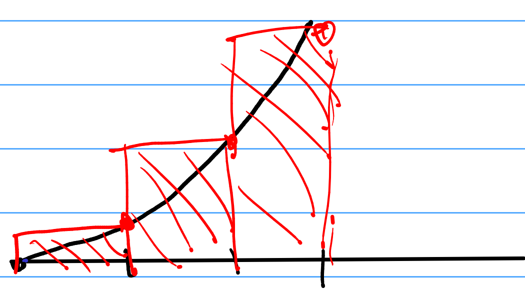


$$R_n = \Delta x \sum_{i=1}^{n} f(a + i\Delta x)$$

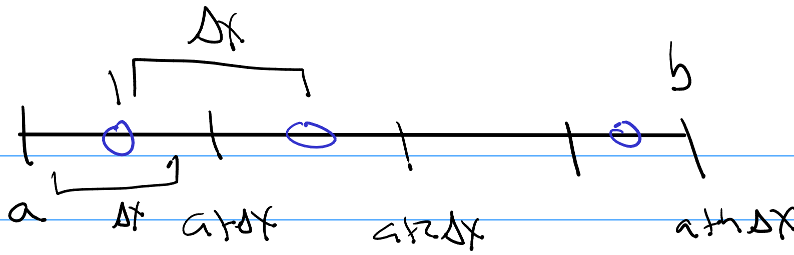
Ex) $R_3 \qquad \Delta x = \frac{3-0}{3} = 1$



$$R_3 = 1\left(1^2 + 2^2 + 3^2\right) = \boxed{14}$$

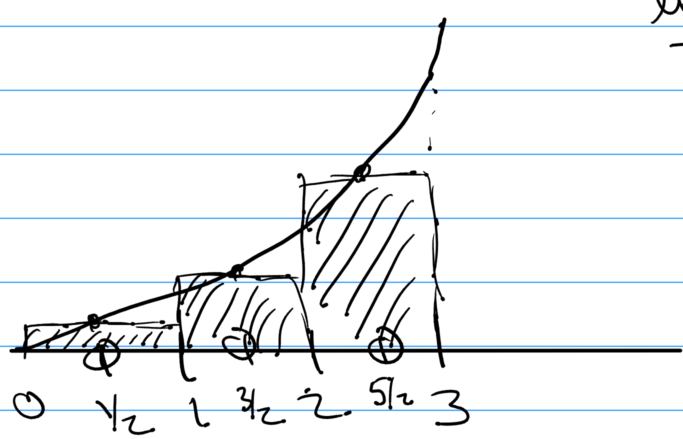


c.) Midpoints



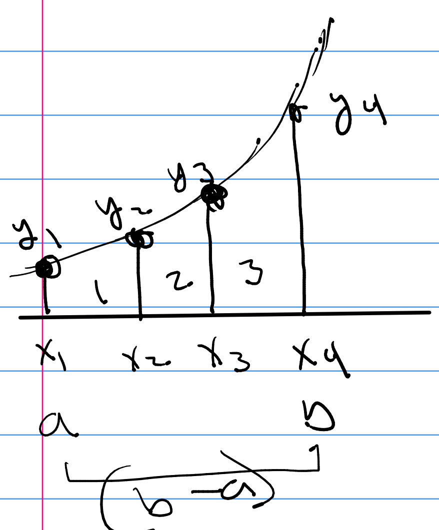
$\Delta x = \frac{b-a}{n}$

$$M_n = \Delta x \sum_{i=0}^{n-1} f\Big(\underbrace{(a+\tfrac{1}{2}\Delta x) + i\Delta x}_{\frac{\text{left} + \text{right}}{2}}\Big)$$



$$M_3 = (1)\left(\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{3}{2}\right)^2 + \left(\tfrac{5}{2}\right)^2\right)$$

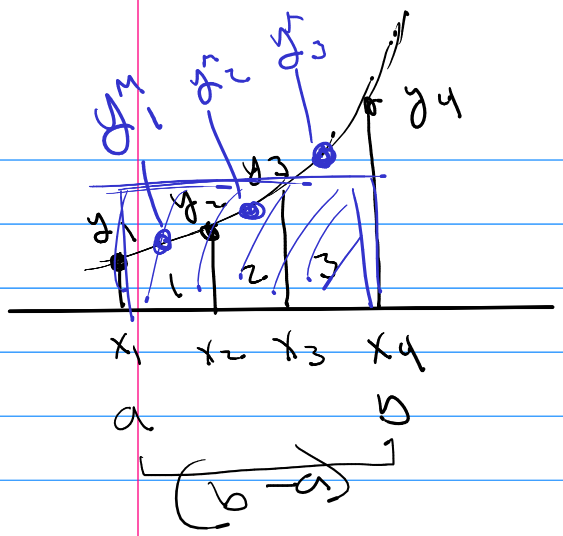
$$= \frac{1}{4}(1+9+25) = \frac{35}{4} = \boxed{8.75}$$



$$L_3 = \frac{b-a}{3}(y_1+y_2+y_3)$$

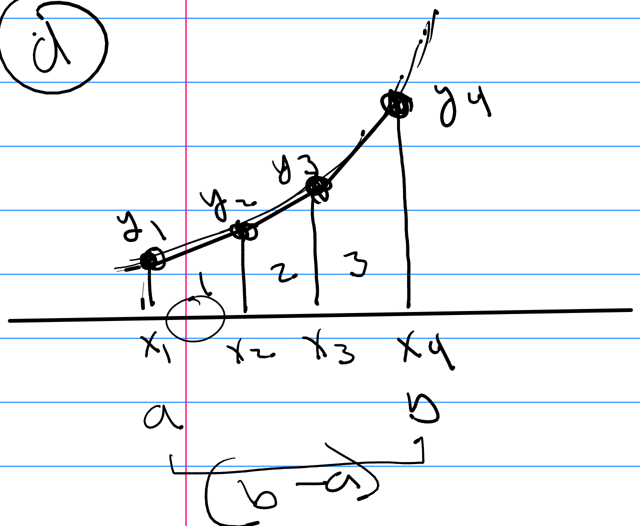
$$L_3 = (b-a)\left(\frac{y_1+y_2+y_3}{3}\right)$$

$$R_3 = \boxed{(b-a)}\boxed{\left(\frac{y_2+y_3+y_4}{3}\right)}$$

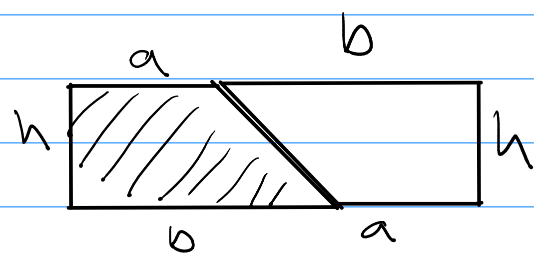


$$M_3 = (b-a)\frac{(y^m_1 + y^m_2 + y^m_3)}{3}$$

(d)



Trapezoidal Approx $= \frac{1}{2}h(a+b)$

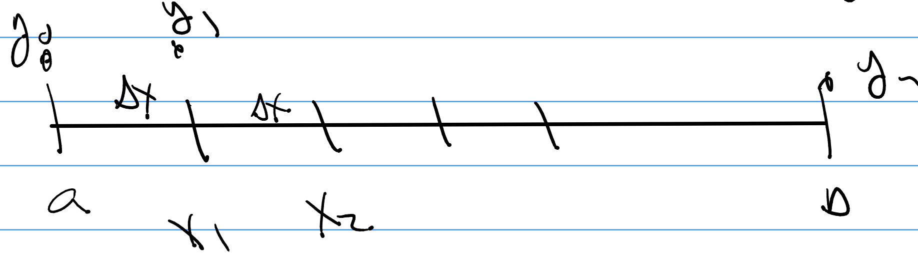


$$T_1 = \frac{1}{2}\Delta x(y_1 + y_2)$$
$$T_2 = \frac{1}{2}\Delta x(y_2 + y_3)$$
$$T_3 = \frac{1}{2}\Delta x(y_3 + y_4)$$

$$\text{Area} \approx T_1 + T_2 + T_3 = \frac{1}{2}\Delta x(y_1 + 2y_2 + 2y_3 + y_4)$$

$$T_n = \frac{1}{2}\Delta x(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$$

$$\Delta x = \frac{b-a}{n}$$



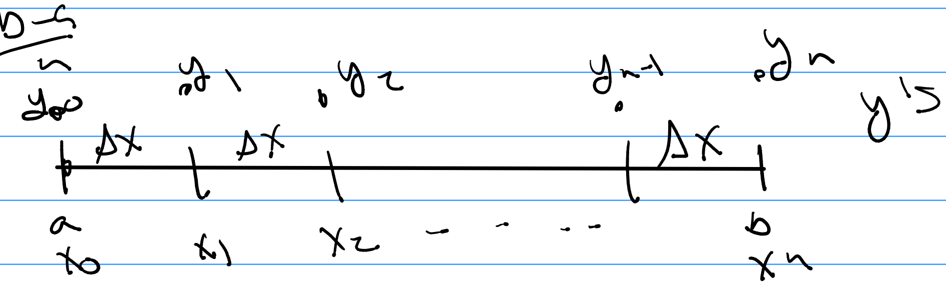
$\boxed{\text{error}}$ if $|f''| \le K$ on $a \le x \le b$

$$|E_T| \le \frac{K(b-a)^3}{12n^2}$$

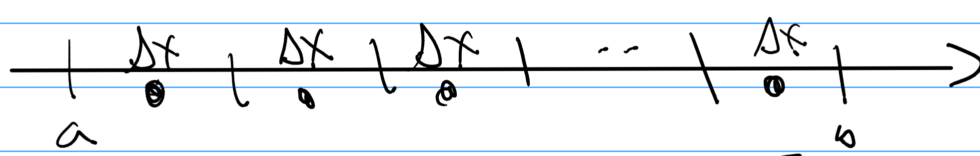
$$|E_M| \le \frac{K(b-a)^3}{24n^2}$$

Finding $\Delta x = \frac{b-a}{n}$

L_n
R_n
T_n
M_n



$$L_n = \Delta x(y_0 + y_1 + \cdots + y_{n-1})$$
$$R_n = \Delta x(y_1 + y_2 + \cdots + y_n)$$
$$T_n = \frac{\Delta x}{2}(y_1 + 2y_2 + 2y_3 + \cdots + 2y_{n-1} + y_n)$$



$\bar{x}_1 \quad \bar{x}_2 \quad \bar{x}_3 \quad \bar{x}_n$

$\bar{x}_1 = a + \frac{1}{2}\Delta x$, $+\Delta x$, $+\Delta x$

$\bar{y}_1 \quad \bar{y}_2 \quad \bar{y}_3 \quad \cdots \quad \bar{y}_n$

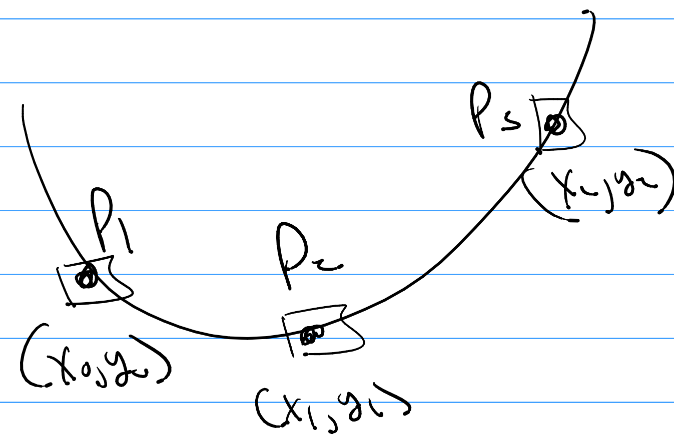
$$M_n = \Delta x(\bar{y}_1 + \bar{y}_2 + \cdots + \bar{y}_n)$$

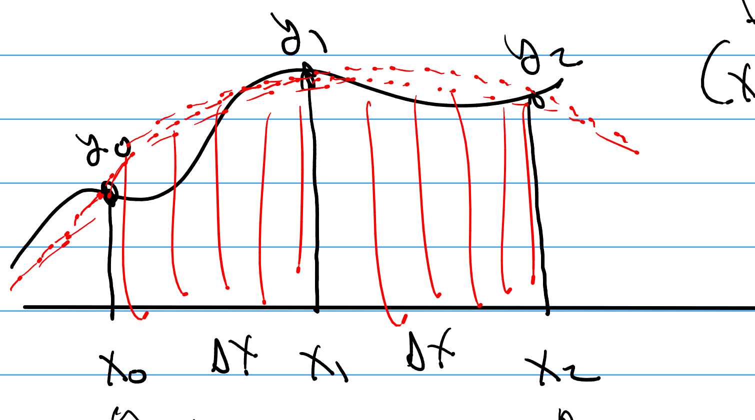
Simpson's Rule

quadratic (parabola)

(1) $\int_d^e (ax^2+bx+c)\,dx = \frac{a}{3}x^3+\frac{b}{2}x^2+cx \Big|_d^e$

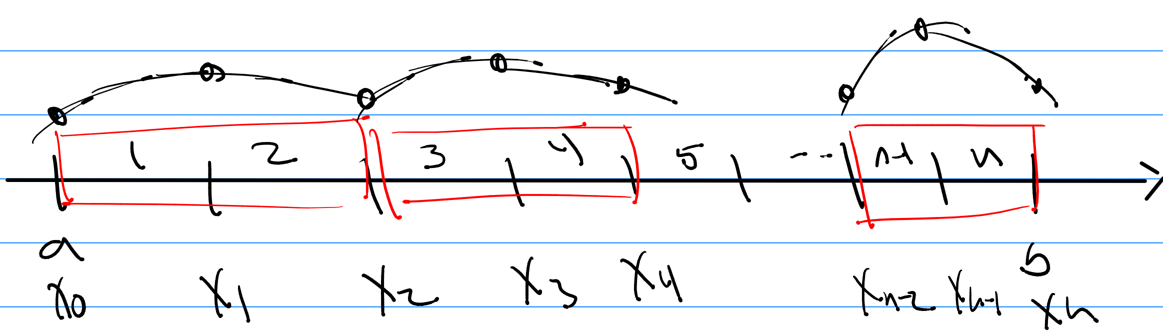
(2) $P = ax^2+bx+c$





→ area under parabola is $\frac{\Delta x}{3}(y_0+4y_1+y_2)$

(1) n must be even



$P_1 = \frac{\Delta x}{3}(y_0+4y_1+y_2)$

$P_2 = \frac{\Delta x}{3}(y_2+4y_3+y_4)$

$P_3 = \frac{\Delta x}{3}(y_4+4y_5+y_6)$ + . . .

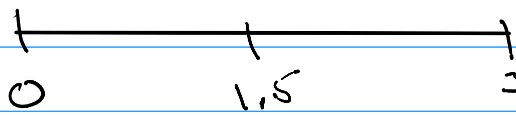
Area = add the parabolas

$$S_n = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

must be even

$$|E_S| \le \frac{K(b-a)^5}{180\, n^4}$$

Ex $\displaystyle\int_0^3 x^2\,dx = 9$ $\Delta x = \frac{3-0}{2} = 1.5$



$\boxed{n=2}$ $S_2 = \frac{\Delta x}{3}\left(y_0 + 4y_1 + y_2\right)$

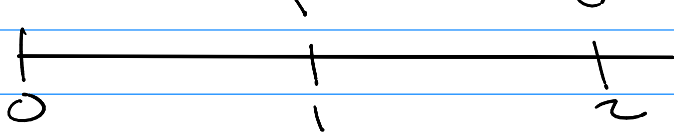
$$S_2 = \frac{1.5}{3}\left(0^2 + 4(1.5)^2 + 3^2\right)$$

$$= \frac{3/2}{3}\left(4(3/2)^2 + 3^2\right)$$

$$= \frac{1}{2}(9+9) = \boxed{9}$$

$\displaystyle\int_0^2 x^3\,dx$ L_2, R_2, T_2, M_2, S_2

$\Delta x = 1$ $0^3 = 0$ $1^3 = 1$ $2^3 = 8$



$L_2 = (1)(0+1) = 1$

$R_2 = (1)(1+8) = 9$

$T_2 = \frac{1}{2}(0 + 2(1) + 8) = 5$

$S_2 = \frac{1}{3}(0 + 4(1) + 8) = 4$

$\int_0^2 x^3 dx = \frac{1}{4}x^4 \Big|_0^2 = \frac{1}{4}2^4 = \boxed{4}$