Math 243

Q5/ $\boxed{\text{trig sub}}$

$\boxed{\text{ex}}$ $$\int_0^1 \frac{dx}{(x^2+1)^2}$$

$$\boxed{\begin{array}{l} (a^2-x^2)^p \rightsquigarrow \text{let } x = a\sin\theta \\ \checkmark\ (a^2+x^2)^p \rightsquigarrow \text{let } x = a\tan\theta \\ (x^2-a^2)^p \rightsquigarrow \text{let } x = a\sec\theta \end{array}}$$

$$\int_0^1 \frac{dx}{(x^2+1)^2}$$ let $x = \tan\theta$

Pyth. identity: $\tan^2\theta + 1 = \sec^2\theta$

$dx = \sec^2\theta\, d\theta$

$$\int_{x=0}^{x=1} \frac{\sec^2\theta\, d\theta}{(\tan^2\theta+1)^2} = \int_{x=0}^{x=1} \frac{\sec^2\theta}{\sec^4\theta}\, d\theta = \int_{x=0}^{x=1} \frac{1}{\sec^2\theta}\, d\theta$$

$$= \int_0^{\pi/4} \cos^2\theta\, d\theta = \int_0^{\pi/4} \left[\frac{1}{2}\cos(2\theta) + \frac{1}{2}\right] d\theta$$

use

$$\cos^2 x = \frac{1}{2}\cos(2x) + \frac{1}{2}$$

$$= \frac{1}{4}\sin(2\theta) + \frac{1}{2}\theta \Big|_0^{\pi/4}$$

$$= \left(\frac{1}{4}\sin(\pi/2) + \pi/8\right) - (0) = \boxed{1/4 + \pi/8}$$

15 Indef. Integral

$$\int \frac{dx}{(x^2+1)^2}$$

Let $x = \tan\theta$

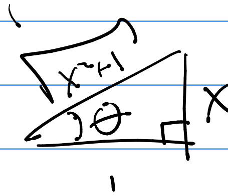
$dx = \sec^2\theta \, d\theta$

$$= \int \frac{\sec^2\theta \, d\theta}{(\tan^2\theta+1)^2} = \int \frac{1}{\sec^2\theta} \, d\theta = \int \cos^2\theta \, d\theta$$

$$= \int \left(\frac{1}{2}\cos 2\theta + \frac{1}{2}\right) d\theta$$

Know

$x = \tan\theta$



$$= \frac{1}{4}\sin(2\theta) + \frac{1}{2}\theta + C$$

know $\sin(2\theta) = 2\sin\theta\cos\theta$

$\theta = \tan^{-1}(x)$

$$= \frac{1}{4}(2)\frac{x}{\sqrt{x^2+1}} \cdot \frac{1}{\sqrt{x^2+1}} + \frac{1}{2}\tan^{-1}x + C$$

$$= \boxed{\frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2}\tan^{-1}x + C}$$

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ex

$$\int e^{2x} \, dx = \frac{1}{2}\int e^u \, du$$

Let $u = 2x$

$du = 2dx$

$$= \frac{1}{2}e^u + C$$

$$= \boxed{\frac{1}{2}e^{2x} + C}$$

ex $\displaystyle\int x e^{2x^2}\,dx = \frac{1}{4}\int e^u\,du$

let $u = 2x^2$ $\quad = \frac{1}{4}e^u + C$

$du = 4x\,dx$

$= \boxed{\frac{1}{4}e^{2x^2} + C}$

ex $\displaystyle\int x e^{2x}\,dx = \int \frac{u}{2} e^u \frac{du}{2}$

let $u = 2x$ $\quad = \frac{1}{4}\boxed{\int (u) e^u\, du}$

$du = 2dx$

by parts

$= \frac{1}{4}\int u e^u\,du = \frac{1}{4}\left[u e^u - \int e^u\,du\right]$

$f(u) = \boxed{u} \xrightarrow{\text{deriv}} f'(u) = 1$

$g'(u) = \boxed{e^u} \xrightarrow{\text{Integrate}} g(u) = e^u$

$= \frac{1}{4}ue^u - \frac{1}{4}e^u + C$

$= \frac{1}{4}(2x)e^{2x} - \frac{1}{4}e^{2x} + C$

$= \boxed{\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C}$

check

deriv $\quad \cancel{\frac{1}{2}e^{2x}} + xe^{2x} - \cancel{\frac{1}{2}e^{2x}} + 0 = xe^{2x}$

ex

$$\int e^x \sin(x)\,dx = -e^x\cos x + \int e^x \cos x\,dx$$

$f = e^x \xrightarrow{\text{deriv}} f' = e^x$

$g' = \sin(x) \xrightarrow{\text{Int.}} g = -\cos x$

$$= -e^x\cos x + \int e^x\cos x\,dx = -e^x\cos x + \left[e^x\sin x - \int e^x \sin x\,dx\right]$$

$f = e^x \xrightarrow{\text{deriv.}} f' = e^x$

$g' = \cos x \xrightarrow{\text{Int}} g = \sin x$

So

$$\int e^x \sin x\,dx = -e^x\cos x + e^x \sin x - \int e^x\sin x\,dx$$

$$2\int e^x \sin x\,dx = e^x\sin x - e^x\cos x + c$$

$$\boxed{\int e^x \sin x\,dx = \tfrac{1}{2}e^x\sin x - \tfrac{1}{2}e^x\cos x + c}$$

check:

$$\left[\tfrac{1}{2}e^x\sin x + \cancel{\tfrac{1}{2}e^x\cos x}\right] - \left[\cancel{\tfrac{1}{2}e^x\cos x} - \tfrac{1}{2}e^x\sin x\right] + 0$$

$= e^x\sin x$!

(5)

$$\int e^x \sin(x)\,dx = -e^x\cos x + \int e^x \cos x\,dx$$

$f = e^x \xrightarrow{\text{deriv}} f' = e^x$

$g' = \sin(x) \xrightarrow{\text{Int}} g = -\cos x$

$$= -e^x\cos x + \int e^x \cos x\,dx = -e^x\cos x + \left[e^x\cos x + \int e^x \sin x\,dx\right]$$

$f = \cos x \xrightarrow{\text{deriv}} f' = -\sin x$

$g' = e^x \xrightarrow{\text{Int}} g = e^x$

$$= \cancel{-e^x\cos x} + \cancel{e^x\cos x} + \int e^x \sin x\,dx$$

$$= \int e^x \sin x\,dx$$

$$\int e^x \sin x\,dx = \int e^x \sin x\,dx \;!$$

So Try something else (See above)

7.4

$$\frac{\text{Poly}}{\text{Poly}} \quad (\text{Rational})$$

ex $$\frac{x^4+2x^3-1}{x^2-x-6}$$

1^{st} Thing $\rightarrow$ Improper Rational ex $\frac{7}{3} = 2 + \frac{1}{3}$

division algorithm

$\boxed{7 = 2 \cdot 3 + 1}$

$\boxed{\frac{7}{3} = 2 + \frac{1}{3}}$

proper fractions

2) Rational Expression

$\frac{P(x)}{Q(x)}$ if degree$(P(x)) \geq$ degree$(Q(x))$

it is improper.

$$\frac{P(x)}{Q(x)} = \underbrace{\boxed{d(x)}}_{\text{poly}} + \frac{r(x)}{Q(x)} \leftarrow \text{proper fraction}$$

ex $\frac{x^4+2x^3-1}{x^2-x-6}$

$$\begin{array}{r} x^2 + 3x + 9 \\ (x^2-x-6)\overline{\big)\, x^4+2x^3+0x^2+0x-1} \\ -(x^4-x^3-6x^2) \\ \hline 3x^3+6x^2+0x \\ -(3x^3-3x^2-18x) \\ \hline 9x^2+18x-1 \\ -(9x^2-9x-54) \\ \hline 27x+53 \end{array}$$

$$= \boxed{x^2+3x+9+\frac{(27x+53)}{(x^2-x-6)}}$$

$$\frac{x^3-2x+1}{x-2}$$

$x-2=0$
$x=2$

$$\begin{array}{c|cccc} 2 & 1 & 0 & -2 & +1 \\ & & 2 & 4 & 4 \\ \hline & 1 & 2 & 2 & 5 \\ & x^2 & x & c & r \end{array}$$

$$\frac{x^3-2x+1}{x-2} = x^2+2x+2+\frac{5}{x-2}$$

$$\boxed{\int \frac{x^3-2x+1}{x-2}\,dx} = \int x^2+2x+2+\frac{5}{x-2}\,dx$$

$$= \boxed{\frac{1}{3}x^3+x^2+2x+5\ln|x-2|+C}$$

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① $\frac{P(x)}{Q(x)}$ and $\deg(P(x)) \geq \deg(Q(x))$

→ use division $\frac{P(x)}{Q(x)} = \underline{d(x)} + \underline{\frac{r(x)}{Q(x)}}$

Pdx

proper rational

Sec 2-5 How to integrate proper rationals!

Idea: $\frac{1}{x} + \frac{1}{y} = \frac{y+x}{xy}$

add by common denom.

$\frac{y+x}{xy} = \frac{1}{x} + \frac{1}{y}$

partial fraction decomposition

ex $\boxed{\frac{1}{x-2} + \frac{1}{x+3}} = \frac{(x+3)+(x-2)}{(x-2)(x+3)}$

?

$= \boxed{\frac{2x+1}{x^2+x-6}}$

Consider $\int \frac{2x+1}{x^2+x-6}\,dx = \int \frac{1}{x-2} + \frac{1}{x+3}\,dx$

$= \ln|x-2| + \ln|x+3| + C$

Need to learn how to decompose proper rationals.

Idea

$$\frac{1}{a \cdot b \cdot c^2} = \frac{?}{a} + \frac{?}{b} + \frac{?}{c} + \frac{?}{c^2} \overset{\text{mult,}}{\underset{\text{common den}}{=}} \frac{?}{a \cdot b \cdot c}$$

factors of denom

Idea

$$\text{polynomial} = (x - r_1)(x - r_2) \cdots (x - r_n)$$

r_i are the polynomials roots.

degree poly = # of roots in the complex plane.

ex $p(x) = (x-2)(x^2+x+2)$

roots $p(x) = 0$

$x - 2 = 0$

$\boxed{x = 2}$

irreducable quadratic

$x^2 + x + 2 = 0$

$$x = \frac{-1 \pm \sqrt{1-8}}{2} = \frac{-1 \pm \sqrt{7}\,i}{2}$$

$$\boxed{x = -\frac{1}{2} + \frac{\sqrt{7}}{2}i} \quad \boxed{x = -\frac{1}{2} - \frac{\sqrt{7}}{2}i}$$

→ any polynomial has as its reduced factors:

$(ax+b)$ — linear (real roots)

(ax^2+bx+c) — irreduc. quadratic (complex roots)

$$x^3 - x^2 - 4 = (x-2)(x^2+x+2)$$

So $\to \displaystyle\int \frac{1}{x^3-x^2-4}\,dx$

but $\dfrac{1}{x^3-x^2-4} = \dfrac{1}{(x-2)(x^2+x+2)} = \dfrac{?}{x-2} + \dfrac{?}{(x^2+x+2)}$

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② $\dfrac{P(x)}{Q(x)}$ in proper rational form

and $Q(x)$ has distinct linear factors

ex $Q(x) = (x+1)(2x-3)(x-4)$

$$\to \frac{P(x)}{Q(x)} = \frac{\text{const}}{\text{linear factor}} + \frac{\text{const}}{\text{linear factor}} + \cdots + \frac{\text{const}}{\text{linear factor}}$$

ex $$\frac{3x+1}{(x+1)(2x-3)(x-4)} = \frac{c_1}{x+1} + \frac{c_2}{2x-3} + \frac{c_3}{x-4}$$

$$= \frac{c_1(2x-3)(x-4) + c_2(x+1)(x-4) + c_3(x+1)(2x-3)}{(x+1)(2x-3)(x-4)}$$

So numerator left = numerator right

$$(3x+1) = c_1(2x^2-11x+12)+c_2(x^2-3x-4) + c_3(2x^2-x-3)$$

$$(3x+1) = (2c_1+c_2+2c_3)x^2 + (-11c_1-3c_2-c_3)x + (12c_1-4c_2-3c_3)$$

$$\boxed{\begin{aligned} x^2 &: & 0 &= 2c_1+c_2+2c_3 \\ x &: & 3 &= -11c_1-3c_2-c_3 \\ \text{const} &: & 1 &= 12c_1-4c_2-3c_3 \end{aligned}}$$

3 eqn's 3 unknowns!

$\to$ use substitution and/or elimination

$c_1 = -8/100$

$c_2 = -88/100$

$c_3 = 52/100$

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(3) $Q(x)$ has repeated linear factors.

ex $Q(x) = (x+1)(x-2)(x-2)(x-2)$

$= (x+1)(x-2)^3$

$$\to \frac{P(x)}{Q(x)} = \frac{c_1}{x+1} + \frac{c_2}{(x-2)^1} + \frac{c_3}{(x-2)^2} + \frac{c_4}{(x-2)^3}$$

(4) $Q(x)$ has distinct irreducible quad.

factor like $(ax^2+bx+c) \to \dfrac{c_1x+c_2}{ax^2+bx+c}$

ex $$\frac{p(x)}{Q(x)} = \frac{p(x)}{(x-2)(x^2+x+3)}$$

$$= \frac{c_1}{x-2} + \frac{c_2x+c_3}{x^2+x+3}$$

(5) $Q(x)$ has non-distinct irred. quad.

ex $(x^2+x+3)^3$

$$\rightsquigarrow \frac{c_1x+c_2}{(x^2+x+3)^1} + \frac{c_3x+c_4}{(x^2+x+3)^2} + \frac{c_5x+c_6}{(x^2+x+3)^3}$$

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ex $$\frac{1}{(x+2)(x-3)^3(x^2+x+1)(x^2+x+2)^2}$$

$$= \frac{c_1}{x+2} + \frac{c_2}{(x-3)^1} + \frac{c_3}{(x-3)^2} + \frac{c_4}{(x-3)^3} + \frac{c_5x+c_6}{x^2+x+1}$$

$$+ \frac{c_7x+c_8}{(x^2+x+2)^1} + \frac{c_9x+c_{10}}{(x^2+x+2)^2}$$

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ex $$\int \frac{x^3+4x^2+x-1}{x^3+x^2}\,dx$$

improper so long divide

$$\begin{array}{r} 1 \\ x^3+x^2+0x+0 \overline{\big) x^3+4x^2+x-1} \\ -(x^3+x^2+0x+0) \\ \hline 3x^2+x-1 \end{array}$$

$$= \int 1 + \frac{3x^2+x-1}{x^3+x^2}\,dx$$

$$= x + \int \frac{3x^2+x-1}{x^3+x^2}\,dx$$

Consider $\dfrac{3x^2+x-1}{x^3+x^2} = \dfrac{3x^2+x-1}{(x)^2(x+1)} = \dfrac{c_1}{x} + \dfrac{c_2}{x^2} + \dfrac{c_3}{(x+1)}$

Numerators $3x^2+x-1 = c_1 x(x+1) + c_2(x+1) + c_3(x^2)$

$$3x^2+x-1 = (c_1+c_3)x^2 + (c_1+c_2)x + (c_2)$$

x^2 : $c_1 + c_3 = 3 \longrightarrow c_3 = 1$

x : $c_1 + c_2 = 1 \rightarrow c_1 = 2$

const : $c_2 = -1$

So $\dfrac{3x^2+x-1}{(x)^2(x+1)} = \boxed{\dfrac{2}{x} + \dfrac{-1}{x^2} + \dfrac{1}{(x+1)}}$

and $= x + \int \boxed{\dfrac{3x^2+x-1}{x^3+x^2}}\,dx$ $\qquad \int x^{-2}\,dx$

$$= x + \int \frac{2}{x}\,dx - \int \frac{1}{x^2}\,dx + \int \frac{1}{x+1}\,dx$$

$$= \boxed{x + 2\ln|x| + \frac{1}{x} + \ln|x+1| + C}$$

Strategy

(1) Recognize? (Tables of Integrals help here)

(2) Simplify? (any $\boxed{\text{identities}}$ or $\boxed{\text{algebra?}}$)

$$\int \frac{x^2-x-6}{x^2-4}\,dx = \int \frac{(x-3)\cancel{(x+2)}}{\cancel{(x+2)}(x-2)}\,dx$$

$$= \int \frac{x-3}{x-2}\,dx$$

(3) Substitution?

$$\int \left[f(g(x))\right]\left[g'(x)\right]dx$$

$\boxed{u = g(x)} \to x = u$

$du = g'(x)\,dx$

$$\int \frac{x}{\sqrt{x-2}}\,dx$$

let $u = x+2 \to u-2 = x$

$du = dx$

$$\int \frac{u-2}{\sqrt{u}}\,du = \int \frac{u}{\sqrt{u}} - \frac{2}{\sqrt{u}}\,du$$

$$= \int (u^{1/2} - 2u^{-1/2})\,du = \ldots$$

(4) Does it meet any of the forms you know?

a) by parts?
b) trig sub?
c) have trig functions?
d) radicals?

→ Rationalization Sub.

$$\text{let } u = (g(x))^{1/n} \to u^n = g(x)$$

$$n u^{n-1} du = g'(x) dx$$

(5) ??? Still no ans ???

a) Try only "weird" substitution
b) Try a different by parts
c) use identities / rationalize denom.
d) Compare to known problems.

Note: Elementary Functions..

→ Function with finite numbers of poly, rational, radical, power, exponential, log, trig / inv, hyper / inv with $+, -, *, \div$, and composition ex $f(x) = \sqrt{\dfrac{x^3+\sinh(x)}{\tanh^{-1}(x^2+1)}} - e^{2x+x^3-\log_{10}(x+1)}$

$\text{deriv}(\text{elementary function}) = \text{elementary function}$

But $\displaystyle\int (\text{elementary function})\, dx$

$= (\,???\,)$ not elementary the majority of the time

ex $\displaystyle\int e^{-x^2}\,dx = \boxed{}$ can not be written as an elem. function

p 517 $\displaystyle\int \frac{e^x}{x}\,dx$, $\displaystyle\int \sin(x^2)\,dx$, $\displaystyle\int \cos(e^x)\,dx$

$\displaystyle\int \frac{1}{\ln x}\,dx$, ...

Other tools?

Hard Integrals (that are elementary)

① Tables of Integrals.

ex $\int y\sqrt{6+4y-4y^2}\,dy$

$$-4y^2+4y+6 = -4\left[\underline{y^2 - y + \left(\tfrac{1}{2}\right)^2} - \left(\tfrac{1}{2}\right)^2\right]+6$$

$$= -4\left[\left(y-\tfrac{1}{2}\right)^2 - \tfrac{1}{4}\right]+6$$

$$= -4\left(y-\tfrac{1}{2}\right)^2 + 7$$

So $\int y\sqrt{6+4y-4y^2}\,dy = \int y\sqrt{7-4\left(y-\tfrac{1}{2}\right)^2}\,dy$

$$= 2\int y\sqrt{7/4-\left(y-\tfrac{1}{2}\right)^2}\,dy$$

let $u = y-\tfrac{1}{2}$ $du = dy$

$$= 2\int \left(u+\tfrac{1}{2}\right)\sqrt{7/4-u^2}\,du$$

$$= 2\int \left(u\sqrt{7/4-u^2}\right)+\left(\tfrac{1}{2}\sqrt{7/4-u^2}\right)du$$

substitution #30

(2) Computer tools (symbolic Math)

Maxima, Mathematica, maple, mupad,

python → symbolic python

SAGE Math