Math 243

Integration

① Recognize a derivative? $\int (\square)\, dx$

② Substitution $\int (a \cdot b)\, dx$

$f(g(x))$ $g'(x)$

let $u = g(x)$

→ Recognize?

③ By Parts $\int (\overset{f}{a} \cdot \overset{g'}{b})\, dx = \boxed{fg} - \int g f'\, dx$

→ do you see a product of two things..

a) $f(x) = \boxed{\text{Something you can differentiate}} \xrightarrow{\text{deriv}} f'(x) = \square$

b) $g'(x) = \boxed{\text{Something you can integrate}} \xrightarrow{\text{Integrate}} g(x) = \square$

ex1 $\int (x-1)(x+2)\, dx = \int (x^2 + x - 2)\, dx$

$= \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + C$

ex $\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{du}{u}$

let $u = \cos x$ $\quad = -\ln|u| + C$

$du = -\sin x\, dx$

$$\int \tan x\,dx = -\ln|u| + C = -\ln|\cos x| + C$$

prop. of ln

$$(\ln ab) = \ln a + \ln b$$

$$\ln a/b = \ln a - \ln b$$

$$c \ln a = \ln a^c$$

$$\boxed{\int \tan x\,dx = -\ln|\cos x| + C = \ln|\sec x| + C}$$

ex

$$\int x \sin x\,dx = \frac{1}{2}x^2 \sin x - \int \frac{1}{2}x^2 \cos x\,dx$$

~~try by parts:~~

$$f(x) = \boxed{\sin x} \rightarrow \text{deriv} \rightarrow f' = \cos x$$

$$g'(x) = x \rightarrow \text{integrate} \rightarrow g = \boxed{\tfrac{1}{2}x^2}$$

try again

$$\int x \sin x\,dx = -x\cos x - \int (1)(-\cos x)\,dx$$

$$f(x) = \boxed{x} \rightarrow \text{deriv} \rightarrow f' = 1$$

$$g'(x) = \sin x \rightarrow \text{Integrate} \rightarrow g = \boxed{-\cos x}$$

$$= -x\cos x + \int \cos x\,dx =$$

$$= \boxed{-x\cos x + \sin x + C}$$

ex $\int (x-1)\sin(\pi x)\,dx = \int \left(x\sin(\pi x) - \sin(\pi x)\right) dx$

$= \int x\sin(\pi x)\,dx - \int \sin(\pi x)\,dx$

#1 #2

do #2 $\int \sin(\pi x)\,dx = \frac{1}{\pi}\int \sin u\,du$

let $u = \pi x$ $\quad = -\frac{1}{\pi}\cos u + C$

$du = \pi\,dx$ $\quad = \boxed{-\frac{1}{\pi}\cos(\pi x) + C}$ #2

do #1 $\int x\sin(\pi x)\,dx = \int \boxed{\frac{u}{\pi}}\sin(u)\boxed{\frac{du}{\pi}}$

let $u = \pi x$

$du = \pi\,dx$ $\quad = \frac{1}{\pi^2}\int u\sin u\,du$

$=$ did integration by parts for this already $= \frac{1}{\pi^2}\left[-u\cos u + \sin u\right] + C$

$= \boxed{\frac{1}{\pi^2}\left(-\pi x\cos(\pi x) + \sin(\pi x)\right) + C}$ #1

so

$\int (x-1)\sin(\pi x)\,dx = \boxed{\#1} - \boxed{\#2}$

$= \left[-\frac{x}{\pi}\cos(\pi x) + \frac{1}{\pi^2}\sin(\pi x)\right] - \left[-\frac{1}{\pi}\cos(\pi x)\right] + C$

$= \boxed{\frac{(1-x)}{\pi}\cdot\cos(\pi x) + \frac{1}{\pi^2}\sin(\pi x) + C}$

$\boxed{\text{ex}}$ $\int \ln(\sqrt{x})\,dx = x\ln\sqrt{x} - \int \frac{1}{2x}\cdot x\,dx$

try by parts. $f(x) = \ln(\sqrt{x}) \xrightarrow{\text{deriv}} f' = \frac{1}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{2x}$

$g'(x) = 1 \xrightarrow{\text{integrate}} g = x$

$\to \int \ln\sqrt{x}\,dx = x\ln\sqrt{x} - \int \frac{1}{2}\,dx$

$= \boxed{x\ln\sqrt{x} - \frac{1}{2}x + C}$

$\boxed{\text{ex}}$ $\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr$? let $u = 4+r^2$

$du = 2r\,dr$

? $\frac{d}{dx}[\sinh^{-1} x] = \frac{1}{\sqrt{1+x^2}}$

$\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr = \int_0^1 (r^3)\left(\frac{1}{\sqrt{4+r^2}}\right)dr$

by parts $f(r) = r^3 \xrightarrow{\text{deriv}} f' = 3r^2$

$g'(r) = \frac{1}{\sqrt{4+r^2}} \xrightarrow{\text{integrate}}$ ⊛ $\boxed{g = \sinh^{-1}(r/2)}$

⊛ $\frac{1}{2}\int \frac{1}{\sqrt{1+(r/2)^2}}\,dr = \int \frac{1}{\sqrt{1+u^2}}\,du = \sinh^{-1} u + C$

let $u = r/2$

$du = \frac{1}{2}dr$

$= \sinh^{-1}(r/2) + C$

$$\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr = r^3 \sinh^{-1}(r/2)\Big|_0^1 - 3\int_0^1 r^2 \sinh^{-1}(r/2)\,dr$$

?

try something else

$$\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr = \frac{1}{2}\int_4^5 \frac{u-4}{\sqrt{u}}\,du = \frac{1}{2}\int_4^5 \frac{u}{\sqrt{u}} - \frac{4}{\sqrt{u}}\,du$$

$$\boxed{\begin{array}{l} \text{let } u = 4+r^2 \rightarrow r^2 = u-4 \\ du = 2r\,dr \end{array}}$$

$$= \frac{1}{2}\int_4^5 \left(u^{1/2} - 4u^{-1/2}\right)du$$

$$= \frac{1}{2}\left[\frac{2}{3}u^{3/2} - 4\cdot 2\cdot u^{1/2}\right]_{u=4}^{u=5}$$

$$= \left[\frac{1}{3}u^{3/2} - 4u^{1/2}\right]_{u=4}^{u=5}$$

$$= \left(\frac{1}{3}5^{3/2} - 4\cdot 5^{1/2}\right) - \left(\underbrace{\frac{1}{3}4^{3/2}}_{8/3} - \underbrace{4\cdot 4^{1/2}}_{8}\right)$$

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ex

$$\int_0^{\pi} e^{\cos t}\sin t\,dt = -\int_1^{-1} e^u\,du = \int_{-1}^{1} e^u\,du$$

let $u = \cos t$

$du = -\sin t\,dt$

$$= e^u\Big|_{u=-1}^{u=1} = \boxed{e - \frac{1}{e}}$$

ex

$$\int_0^{\pi} e^{\cos t}\sin(2t)\,dt$$

$$\int_0^{\pi} e^{\cos t}\sin(2t)\,dt = \frac{1}{2}\int_0^{2\pi} e^{\cos(\frac{1}{2}u)}\sin u\,du \;?$$

$u = 2t \Rightarrow t = \frac{1}{2}u$

try $\boxed{\text{sub}}$ $du = 2\,dt$

try by parts $\displaystyle\int_0^{\pi} e^{\cos t}\sin(2t)\,dt = -\frac{1}{2}e^{\cos t}\cos 2t\Big|_0^{\pi} - \frac{1}{2}\int_0^{\pi} e^{\cos t}\sin t\cos 2t\,dt$

$f(x) = e^{\cos t} \to \text{deriv} \to f' = -e^{\cos t}\sin t$

$g'(x) = \sin(2t) \to \text{integrate} \to g = \frac{-1}{2}\cos(2t)$

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try $\sin(2t) = 2\sin t\cos t$

$$\to 2\int_0^{\pi} e^{\cos t}\cos t\sin t\,dt = -2\int_{u=1}^{u=-1} u e^u\,du$$

let $u = \cos t$

$du = (-)\sin t\,dt$

$$= 2\int_{-1}^{1} (u)(e^u)\,du = 2\left[ue^u\Big|_{-1}^{1} - \int_{-1}^{1} e^u\,du\right]$$

by parts

$f(u) = u \xrightarrow{\text{deriv}} f' = 1$

$g'(u) = e^u \xrightarrow{\text{int.}} g = e^u$

$$= 2\left[ue^u - e^u\right]_{-1}^{1}$$

$$= 2\left[(0) + \left(\frac{+1}{e} + \frac{1}{e}\right)\right] = \boxed{\frac{4}{e}}$$

$\cos^2 x + \sin^2 x = 1 \to \sin^2 x = \boxed{1 - \cos^2 x}$

$$\int \frac{1}{(\cos^2 x + \sin^2 x)^{3/2}}\,dx = \int 1\,dx$$

$$\frac{3x^4}{\sqrt{1-x^2}}$$

7.2 Trig Integrals

$\int \sin^m x \cos^n x\,dx$

$\int \tan^m x \sec^n x\,dx$

Know $\to \sin^2 x + \cos^2 x = 1$ $\qquad \sin(2x) = 2\sin x \cos x$

$\tan^2 x + 1 = \sec^2 x$

$\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - (1 - \cos^2 x)$

$\cos 2x = 2\cos^2 x - 1 \to \cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$\cos 2x = 1 - 2\sin^2 x \to \sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\int \sin x\,dx = -\cos x + C$ $\qquad \int \sec x \tan x\,dx = \sec x + C$

$\int \cos x\,dx = \sin x + C$ $\qquad \int \sec^2 x\,dx = \tan x + C$

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ex $\int \sin^2 x\,dx = \frac{1}{2}\int 1 - \cos 2x\,dx$

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

ex

$$\int \sin^3 x \cos^3 x\, dx = \int u^3 \cos^2 x\, du$$

let $u = \sin x$ b/c $\cos^2 x = \underbrace{1 - \sin^2 x}_{= 1 - u^2}$

$du = \cos x\, dx$

$$= \int u^3 (1 - u^2)\, du = \int u^3 - u^5\, du$$

= finish

$$\int \sin^m x \cos^n x\, dx$$

① odd # of $\cos(x)$

let $u = \sin x$ $\rightarrow$ $\int$ (poly of u) du

$du = \cos x\, dx$

plus $\cos^2 x = 1 - \sin^2 x$

$= 1 - u^2$

② odd # of $\sin(x)$

let $u = \cos x$ $\rightarrow$ $\int$ (poly of u) du

$du = -\sin x\, dx$

plus: $\sin^2 x = 1 - \cos^2 x$

$= 1 - u^2$

③ both even #'s of $\sin(x)$ $\cos(x)$

Use identities: ① $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$

$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$

② $\sin x \cos x = \frac{1}{2}\sin(2x)$

know $\frac{d}{dx}[\tan x] = \sec^2 x$

$\frac{d}{dx}[\sec x] = \sec x \tan x$

$1 + \tan^2 x = \sec^2 x$

$$\int \tan^m x \sec^n x \, dx$$

(1) even # of sec(x)

let $u = \tan x$
$du = \sec^2 x \, dx$ $\rightarrow \int (\text{poly of } u) \, du$

plus: $\sec^2 x = 1 + \tan^2 x$
$= 1 + u^2$

(2) odd # of tan(x)

let $u = \sec x$
$du = \sec x \tan x \, dx$ $\rightarrow \int (\text{poly of } u) \, du$

plus: $\tan^2 x = \sec^2 x - 1$
$= u^2 - 1$

(3) odd # of sec
even # of tan

$\boxed{???}$

be creative $\rightarrow$ know $\int \tan x \, dx = \ln|\sec x| + C$

know $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

$\rightarrow$ by parts?

$\rightarrow$ sub. and by parts?

ex) $\int \sin^3 x \cos^4 x \, dx$

$= \int \sin^2 x \cos^4 x \, (\sin x \, dx) = -\int \sin^2 x \; u^4 \, du$

$\sin^2 x = 1-u^2$

let $u = \cos x$

$du = -\sin x \, dx$

plus: $\sin^2 x = 1-\cos^2 x$

$= 1-u^2$

$= -\int (1-u^2) u^4 \, du$

$= \int u^6 - u^4 \, du = \frac{1}{7}u^7 - \frac{1}{5}u^5 + C$

$= \boxed{\frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C}$

ex) $\int \sin^3 x \cos^7 x \, dx = \int \sin^3 x \cos^6 x \, (\cos x \, dx)$

let $u = \sin x$

$du = \cos x \, dx$

plus: $\cos^2 x = 1-\sin^2 x$

$= 1-u^2$

$= \int u^3 \cos^6 x \, du$

$\cos^6 x = (\cos^2 x)^3 = (1-u^2)^3$

$= \int u^3 (1-u^2)^3 \, du$

know $(a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

$(1-u^2)^3 = 1 - 3u^2 + 3u^4 - u^6$

$= \int (u^3 - 3u^5 + 3u^7 - u^9) \, du$

$$= \frac{1}{4}u^4 - \frac{1}{2}u^6 + \frac{3}{8}u^8 - \frac{1}{10}u^{10} + C$$

$$\boxed{= \frac{1}{4}\sin^4 x - \frac{1}{2}\sin^6 x + \frac{3}{8}\sin^8 x - \frac{1}{10}\sin^{10} x + C}$$

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ex

$$\int \tan^2 x \sec^6 x \, dx = \int \tan^2 x \sec^4 x \left(\sec^2 x \, dx\right)$$

let $u = \tan x$

$du = \sec^2 x \, dx$

plus $\sec^2 x = 1 + \tan^2 x$

$= 1 + u^2$

$$= \int u^2 \sec^4 x \, du$$

$\sec^4 x = (1+\tan^2 x)^2$

$$= \int u^2 (1+u^2)^2 \, du$$

$$= \int u^2 (1 + 2u^2 + u^4) \, du$$

$$= \int (u^2 + 2u^4 + u^6) \, du = \frac{1}{3}u^3 + \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$$

$$= \boxed{\frac{1}{3}\tan^3 x + \frac{2}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C}$$

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ex

$$\int \sec x \, dx = \int \sec x \, \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

let $u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x) \, dx$

$$= \int \frac{1}{u} \, du$$

$$= \ln|u| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

7.3 Trigonometric Substitution

$$\int \sqrt{1+x^2}\, dx \qquad 1+\tan^2\theta = \sec^2\theta$$

let $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

$$\int \sqrt{1+\tan^2\theta}\, \sec^2\theta \, d\theta$$

$$= \int \sqrt{\sec^2\theta}\, \sec^2\theta \, d\theta$$

$$= \int \sec^3\theta \, d\theta$$

Use $x =$ trig function substitution?

(1) $\sqrt{a^2-x^2}$ let $x = a\sin\theta$ $-\pi/2 \le \theta \le \pi/2$

$\hookrightarrow \sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} = a\sqrt{1-\sin^2\theta}$

$= a\sqrt{\cos^2\theta} = a\cos\theta$

(2) $\sqrt{a^2+x^2}$ let $x = a\tan\theta$ $\quad -\pi/2 < \theta < \pi/2$

$$\sqrt{a^2+x^2} = \sqrt{a^2 + a^2\tan^2\theta}$$
$$= a\sqrt{1+\tan^2\theta} = a\sec\theta$$

(3) $\sqrt{x^2-a^2}$ let $x = a\sec\theta$ $\quad 0 \le \theta < \pi/2$
or $\pi \le \theta < 3\pi/2$

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ex $\displaystyle\int \frac{dx}{\sqrt{1+x^2}}$ $\qquad \frac{d}{dx}\left[\sinh^{-1}x\right] = \frac{1}{\sqrt{1+x^2}}$

$$= \sinh^{-1}(x)$$

ex $$\int \frac{x}{\sqrt{1+x^2}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\int u^{-1/2}\,du$$

$\boxed{\text{let } u = 1+x^2 \quad du = 2x\,dx}$

$$= u^{1/2} + C$$
$$= \boxed{\sqrt{1+x^2} + C}$$

ex $$\int \frac{x^3}{\sqrt{4+x^2}}\,dx = \frac{1}{2}\int \frac{(u-4)\,du}{\sqrt{u}} = \frac{1}{2}\int u^{1/2} - 4u^{-1/2}\,du$$

$\boxed{\begin{array}{l}\text{let } u = 4+x^2 \\ du = 2x\,dx\end{array}}$

$$= \frac{1}{2}\left[\frac{2}{3}u^{3/2} - 8u^{1/2}\right] + C$$
$$= \frac{1}{3}(4+x^2)^{3/2} - 4(4+x^2)^{1/2} + C$$

ex) $\displaystyle\int \frac{x^3}{\sqrt{4+x^2}}\,dx = \int \frac{(2\tan\theta)^3}{\sqrt{4+4\tan^2\theta}}\, 2\sec^2\theta\, d\theta$

type $\sqrt{a^2+x^2}$
let $x = a\tan\theta$

let $x = 2\tan\theta$
$dx = 2\sec^2\theta$

$$= 8\int \frac{\tan^3\theta \sec^2\theta\, d\theta}{\sec\theta} = 8\int \tan^3\theta \sec\theta\, d\theta$$

$$= 8\int \boxed{\tan^2\theta}\,\boxed{(\sec\theta\tan\theta)\,d\theta} = 8\int (u^2-1)\,du$$

let $u = \sec\theta$
$du = \sec\theta\tan\theta\, d\theta$
plus: $\tan^2\theta = \sec^2\theta - 1$
$= u^2 - 1$

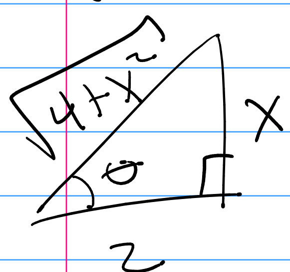
$$= 8\left[\tfrac{1}{3}u^3 - u\right] + C$$

back sub.

$$= \frac{8}{3}\sec^3\theta - 8\sec\theta + C$$

$x = 2\tan\theta$

$\tan\theta = \frac{x}{2}$



$\rightarrow \sec\theta = \dfrac{\sqrt{4+x^2}}{2}$

$$= \frac{8}{3}\left(\frac{\sqrt{4+x^2}}{2}\right)^3 - 8\left(\frac{\sqrt{4+x^2}}{2}\right) + C$$

$$= \frac{1}{3}(4+x^2)^{3/2} - 4(4+x^2)^{1/2} + C$$

ex $\int_0^{1/2} x\sqrt{1-4x^2}\,dx = -\frac{1}{8}\int_1^0 \sqrt{u}\,du$

let $u = 1-4x^2$

$du = -8x\,dx$

$= \frac{1}{8}\int_0^1 u^{1/2}\,du$

$= \frac{1}{8}\frac{2}{3}u^{3/2}\Big|_0^1 = \frac{1}{12}u^{3/2}\Big|_0^1 = \boxed{\frac{1}{12}}$

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ex $\int_0^a x^2\sqrt{a^2-x^2}\,dx = \int_0^{\pi/2} a^2\sin^2\theta\; a\cos\theta\; a\cos\theta\,d\theta$

type $\sqrt{a^2-x^2}$

let $x = a\sin\theta$

$dx = a\cos\theta\,d\theta$

$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} = a\sqrt{1-\sin^2\theta} = a\cos\theta$

$= a^4\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$

$= \frac{a^4}{4}\int_0^{\pi/2}\sin^2(2\theta)\,d\theta$

know $\frac{1}{4}\sin^2 2\theta = \sin^2\theta\cos^2\theta$ $\quad \sin^2\phi = \frac{1}{2}(1-\cos 2\phi)$

$= \frac{a^2}{4}\int_0^{\pi/2}\left(\frac{1}{2} - \frac{1}{2}\cos(4\theta)\right)d\theta$

$= \frac{a^2}{4}\left(\frac{1}{2}\theta - \frac{1}{8}\sin 4\theta\right)\Big|_0^{\pi/2}$

$= \frac{a^2}{4}\frac{\pi}{4} = \boxed{\frac{a^2\pi}{16}}$