Math 243

Q's of exponential problems...

$a' \propto a \rightarrow a' = Ka$

$\therefore \ a(t) = a_0 e^{Kt} \qquad a_0 = a(0)$

@ $\boxed{t=0} \Rightarrow a_0 = 100$

Word Problem: you have $\boxed{100 \text{ units in 1hr.}}$

Wait another 2hrs and you have 2.31 units

$t=2$ $\qquad a(2)=2.31$

Q1 what is growth constant?

$$a(t) = 100e^{Kt}$$

$$\Rightarrow 2.31 = 100e^{K(2)} \Rightarrow K = \boxed{\tfrac{1}{2}\ln(2.31)}$$

$$2.31 = e^{2K}$$

$$\ln(2.31) = 2K$$

Q2 Function?

$$\boxed{a(t) = 100e^{\frac{1}{2}\ln(2.31)\,t}}$$

Q3 What was amount at <u>very</u> beginning?

$$a(-1) = \boxed{100e^{-\frac{1}{2}\ln(2.31)}} =$$

(15) $a(t) = \boxed{a_0} e^{\boxed{k}t}$

$\boxed{t=0 \quad a_0 = ?}$ $\quad t=1 \quad a(1)=100 \quad t=3 \quad a(3)=231$

$$\begin{cases} 100 = a_0 e^{k} \\ 231 = a_0 e^{3k} \end{cases} \quad a_0 = \boxed{100 e^{-k}}$$

$$\Rightarrow 231 = (100 e^{-k})(e^{3k})$$

$$231 = 100 e^{2k}$$

$$\Leftrightarrow$$

$$k = \frac{1}{2}\ln(2.31)$$

$$a_0 = 100 e^{-\frac{1}{2}\ln(2.31)}$$

$\frac{1}{2}$ life $\quad$ $k>0$ $\quad$ doubling time

$a(t) = a_0 e^{-kt}$ $\quad$ $a(t) = a_0 e^{kt}$

$\frac{1}{2}\cancel{a_0} = \cancel{a_0} e^{-k t_h}$ $\quad$ $2\cancel{a_0} = \cancel{a_0} e^{k t_d}$

$\ln(1/2) = -k t_h$ $\quad$ $\ln(2) = k t_d$

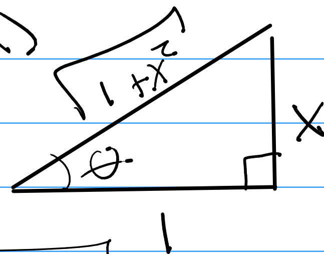
$-\frac{1}{k}\ln(1/2) = t_h$ $\quad$ $\boxed{\frac{1}{k}\ln(2) = t_d}$

$\boxed{\frac{1}{k}\ln(2) = t_h}$

Ex $\quad \sin(\tan^{-1}(x))$

$\theta = \tan^{-1}(x)$

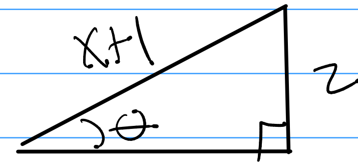
$\tan(\theta) = \dfrac{x}{1}$



$$\boxed{\sin(\tan^{-1}x) = \frac{x}{\sqrt{1+x^2}}}$$

idea:

$\sin\theta = \dfrac{2}{x+1}$



$\sqrt{(x+1)^2 - 2^2} = \sqrt{x^2+2x-3}$

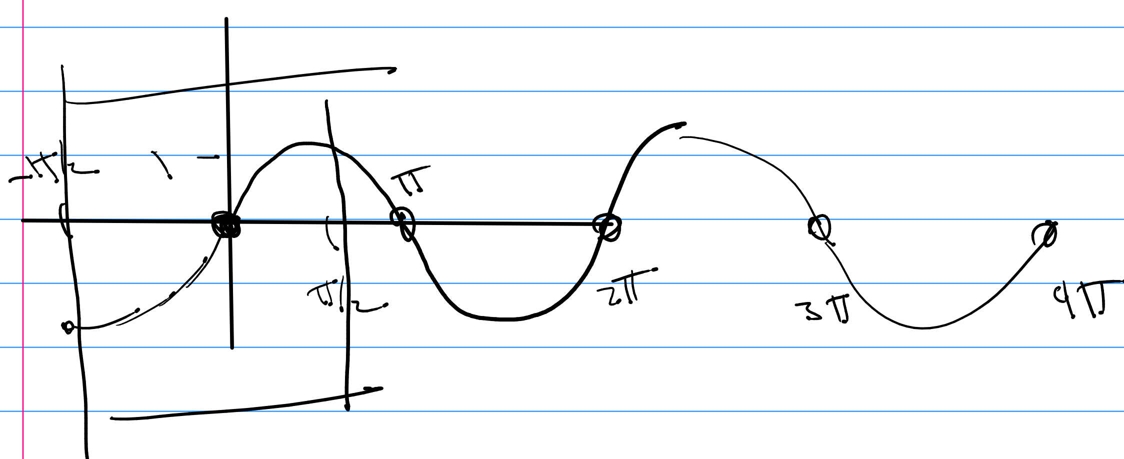
$= \sqrt{(x+3)(x-1)}$

$\tan\theta = \dfrac{2}{\sqrt{(x+3)(x-1)}}$

$\cos\theta = \dfrac{\sqrt{(x+3)(x-1)}}{x+1}$

$\sin^{-1}(\sin(x)) = x \quad \boxed{-\pi/2 \le x \le \pi/2}$

$\sin^{-1}(\sin(4\pi)) = \sin^{-1}(0) = 0$



Ex

uate the integral.

$$\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{4}{1+x^2}\,dx = 4\left[\int \frac{1}{1+x^2}\,dx\right]\Bigg|_{x=1/\sqrt{3}}^{x=\sqrt{3}}$$

$$= 4 \left[\tan^{-1}(x) \right] \Big|_{x=1/\sqrt{3}}^{x=\sqrt{3}}$$

$$\boxed{= 4 \left[\tan^{-1}(\sqrt{3}) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \right]}$$

$$\frac{d}{dx}\left[e^{(3x)} \right] = e^{(3x)} \cdot (3x)'$$

$$= \boxed{3e^{3x}}$$

$$\frac{d}{dx}\left[\tan^{-1}(3x) \right] = \frac{1}{1+(3x)^2} \cdot (3x)'$$

$$= \boxed{\frac{3}{1+9x^2}}$$

$$\frac{d}{dx}\left[\tan^{-1}\left(\frac{3x}{\sin(x)} \right) \right]$$

$$= \frac{1}{1+\left(\frac{3x}{\sin x}\right)^2} \cdot \left(\frac{3x}{\sin(x)} \right)'$$

$$= \boxed{\frac{1}{1+\left(\frac{3x}{\sin(x)}\right)^2}\cdot\left[\frac{(3)(\sin(x)) - (3x)(\cos x)}{\sin^2 x}\right]}$$

Indet. Forms $\boxed{\frac{0}{0}}$, $\frac{\infty}{\infty}$, $\infty - \infty = \frac{0}{0}$ or $\frac{\infty}{\infty}$

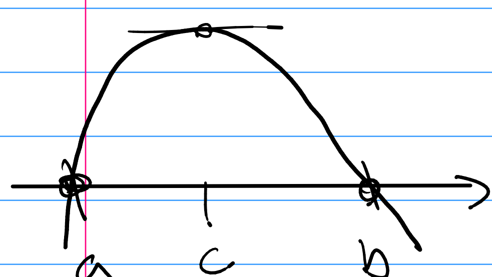
0^0, 1^∞, ∞^0 $\quad f^g = e^{g \cdot \ln f}$

$0 \cdot \infty$

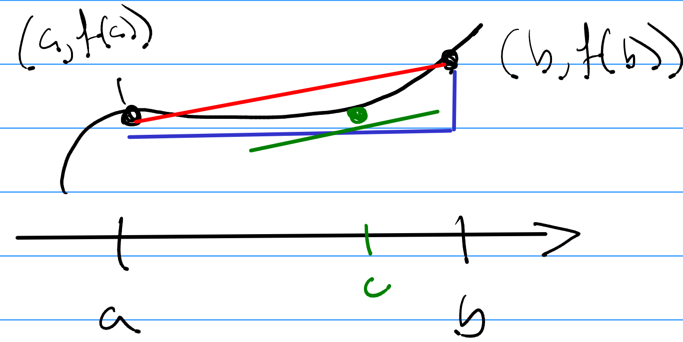
L'Hospital's Rule $\quad \lim \frac{f'(x)}{g'(x)} = L$ and $\lim f = 0$

$\lim g = 0$

$\Rightarrow \lim \frac{f(x)}{g(x)} = L$ $\quad$ Why?



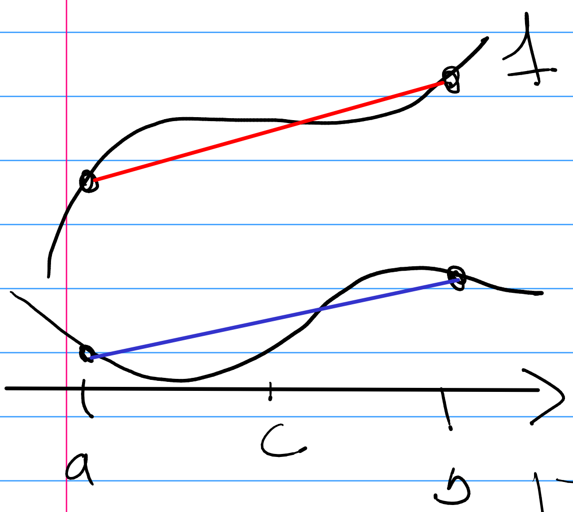
Rolle's thm



$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Mean Value thm

Cauchy's Mean Value Thm



$f \to$ slope $= \dfrac{f(b)-f(a)}{b-a}$

$g \to$ slope $= \dfrac{g(b)-g(a)}{b-a}$

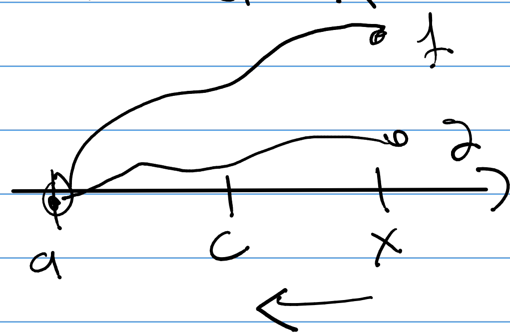
$$\to \frac{\text{slope}}{\text{slope}} = \boxed{\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}}$$

Thm if $\lim\limits_{x\to a} f(x) = \underline{0}$, $\lim\limits_{x\to a} g(x) = \underline{0}$, $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)} = L$

f, g are cont and differentiable

<u>then</u> $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = L$

pf consider $x \to a^+$



<u>Cauchy gives</u> $\dfrac{f(x) - \cancel{f(a)}^{0}}{g(x) - \cancel{g(a)}^{0}} = \dfrac{f'(c)}{g'(c)}$

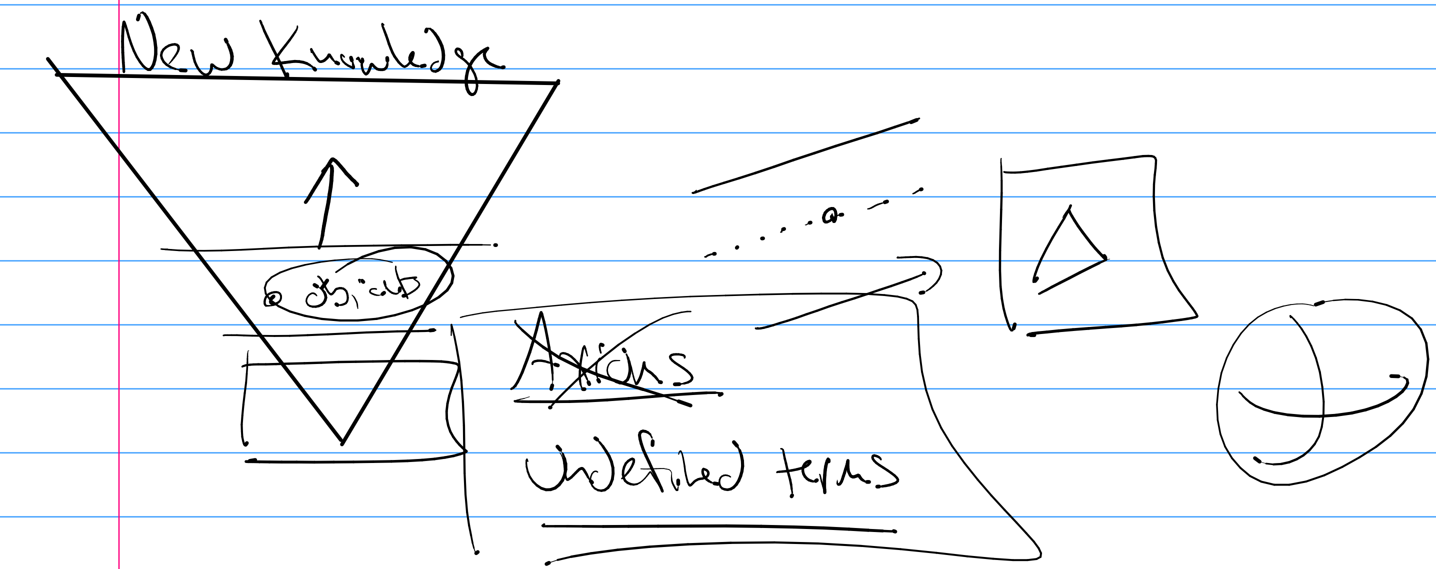
let $F(x) = \begin{cases} f(x) & x \neq a \\ 0 & x = a \end{cases}$

$G(x) = \begin{cases} g(x) & x \neq a \\ 0 & x = a \end{cases}$

By Cauchy $\quad \dfrac{F'(c)}{G'(c)} = \dfrac{F(x) - F(a)}{G(x) - G(a)} = \dfrac{F(x)}{G(x)}$

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{F(x)}{G(x)} = \lim_{x \to a^+} \frac{F'(c)}{G'(c)}$$

$$= \lim_{c \to a^+} \frac{F'(c)}{G'(c)} = \lim_{c \to a^+} \frac{f'(c)}{g'(c)} = L$$



$\boxed{\text{Exam 1}}$ $\boxed{\text{ch 12 / ch 6}}$ $\boxed{\text{1hr 15mins}}$

17 probs @ 10 pts each → 160 pts = 100%

$\boxed{\text{ch 12}}$

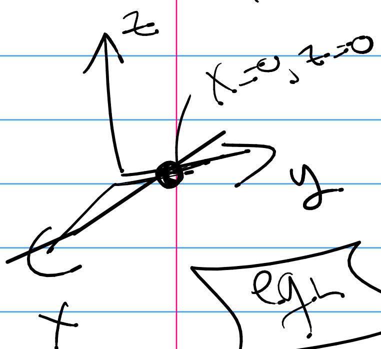
12.1 3D $\boxed{\text{2 probs}}$

① given info (points, lines, algebra)

→ ans: eqn of the sphere

$\boxed{\text{ex}}$ center is @ intersection of $\boxed{x+y=2}$ and y-axis.

radius is same as distance from $(1,1,1)$ and $(2,3,4)$



$\boxed{\text{eqn}}$ center $(0,2,0)$

$$r = (1^2 + 2^2 + 3^2)^{1/2} = \sqrt{14}$$

$$(x-0)^2 + (y-2)^2 + (z-0)^2 = (\sqrt{14})^2$$

② algebraic expression → circle's → center? → radius?

$$x^2 - 2x + y^2 + z^2 + z = 3$$

⇓ Algebra

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = (r)^2$$

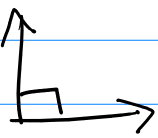
$\boxed{12.2 - 12.4}$ Vectors – $\boxed{6 \text{ probs}}$

① (parts) given $\vec{v} = \langle v_1, v_2, v_3 \rangle = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$

$\vec{u}$

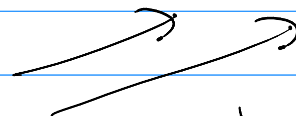
→ $\boxed{\text{do ops}}$

② q's about is $\vec{v} \parallel \vec{u}$? $\vec{v} \perp \vec{u}$?



$a \cdot b = 0$

$|a \times b| = |a||b|$



$a \cdot b = |a||b|$

$|a \times b| = 0$

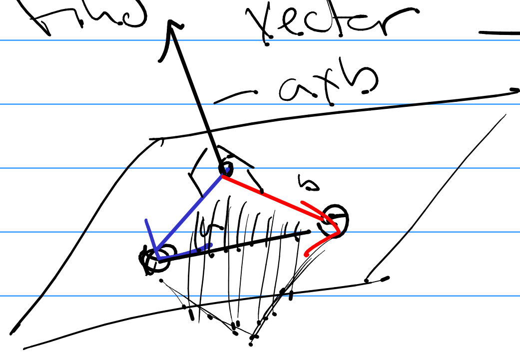
formula

$\boxed{a \cdot b = |a||b|\cos\theta}$

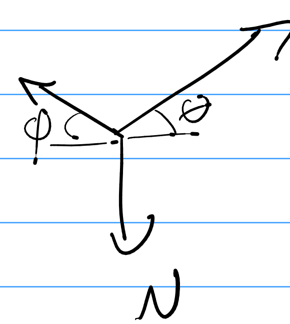
$\boxed{|a \times b| = |a||b|\sin\theta}$

③ give you points on a plane

find vector $\perp$ to plane



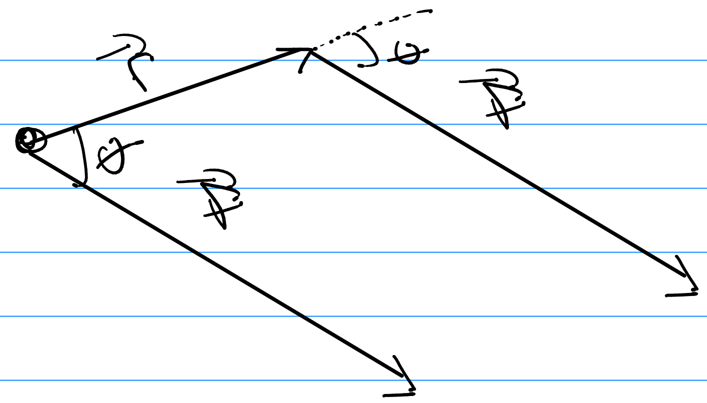
④ Word Problem; Forces



⑤ Word Problem: Work

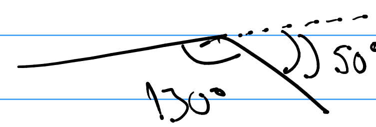
$$\text{Work} = \vec{F} \cdot \vec{D}$$

⑥ Word Problem : Torque



$$|T| = |\vec{r} \times \vec{F}|$$

$$= |\vec{r}||\vec{F}|\sin\theta$$



Ch 6 — 6.5 — 1 prob

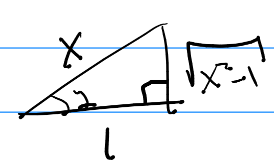
① Word Prob — Growth or Decay

6.6 — Inverse Trig

① right triangle properties & inverse triangle

② derive $\frac{d}{dx}[\sin^{-1}x]$ or $\frac{d}{dx}[\sec^{-1}x]$ or $\frac{d}{dx}[\tan^{-1}x]$

$\frac{dy}{dx} = ?$ $y = \sec^{-1}x \longrightarrow \boxed{\sec y = \frac{x}{1}}$



Implicit Deriv. $\sec y \tan y \frac{dy}{dx} = 1$

so $\frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{x\sqrt{x^2-1}}$

③ Know the derivatives of "everything" plus

$\boxed{\sin^{-1}, \sec^{-1}, \tan^{-1}}$

(parts) doing derivatives

Note one will be a L'Hopital Rule

ex $\frac{d}{dx}\left[\tan^{-1}\left(\frac{x}{x+1}\right)\right]$

ex $\lim_{x\to 0} \frac{\sin^{-1}x}{\sin x} = \lim_{x\to 0} \frac{\frac{d}{dx}[\sin^{-1}x]}{\frac{d}{dx}[\sin x]} = \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{\cos x} = 1$

④ Integrals

→ 4a) recognition → $\int x^2+1+\frac{1}{1+x^2}\,dx$

→ 4b) use substitution

$\frac{1}{3}x^3 + x + \tan^{-1}x + C$

ex $\int \frac{2x}{1+x^2}\,dx = \int \frac{du}{u} = \ln|u|+C$

$= \ln|1+x^2|+C$

ln $u = 1+x^2$

$du = 2x\,dx$

$\boxed{6.7}$ hyperbolics

① Know deriv. of hyp. functions → $\boxed{\begin{matrix}(\sinh x)' \\ (\text{sech} x)' \\ (\tanh x)'\end{matrix}}$

→ (parts) doing Derivatives.

② Derive $\boxed{\frac{d}{dx}[\sinh^{-1} x]}$, $\boxed{\frac{d}{dx}[\tanh^{-1} x]}$

$y = \sinh^{-1} x \xrightarrow[\text{implicit}]{} \underline{\sinh y = x}$

$\frac{dy}{dx} = ?$ $\cosh y \cdot y' = 1$

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \boxed{\frac{1}{\sqrt{1+x^2}}}$$

$\cosh^2 y - \sinh^2 y = 1 \rightsquigarrow \cosh y = \sqrt{1 + \sinh^2 y}$

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③ Do Derivatives with 'everything'

parts

look above

④ Integrals

4a) Recognize? $\int \frac{1}{\sqrt{1+x^2}} dx = \sinh^{-1} x + C$

4b) Substitution

$$\frac{d}{dx}\left[\frac{x+1}{\sqrt{x}} + \cancel{\sinh^{-1}}(2x) - \tan(x)\right]$$

$$= \boxed{\frac{(1)(\sqrt{x}) - (x+1)\frac{1}{2}x^{-1/2}}{x} + \frac{1}{\sqrt{1+(2x)^2}} \cdot 2 - \sec^2 x}$$

$$\sinh(x) = \boxed{\frac{e^x - e^{-x}}{2}}$$