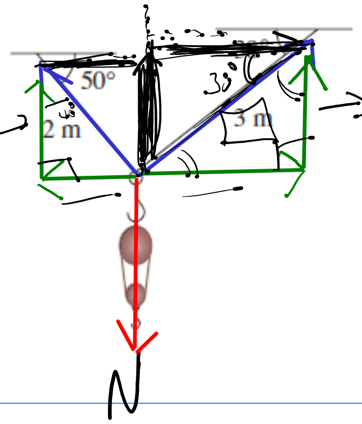
Math 243

~~Q's~~

T_{2y}



T_{3y}

degrees vs radians

$$T_{2y} + T_{3y} = N$$

$$T_{2x} + T_{3x} = 0$$

$$a = \langle 1, -4, 1 \rangle \qquad \langle 0, 7, -7 \rangle = b$$

$$a \cdot b = 1 \cdot 0 + (-4)(7) + (1)(-7)$$
$$= -35$$

$$a \cdot b = |a|\,|b| \cos\theta$$

$$\cos\theta = \frac{a \cdot b}{|a|\,|b|} = \frac{-35}{\sqrt{18}\sqrt{98}} \to \boxed{\theta = \cos^{-1}\left(\frac{-35}{\sqrt{18}\sqrt{98}}\right)}$$

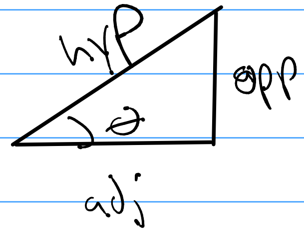
$$x \text{ rad} = x \cancel{\text{rad}} \left(\frac{180^\circ}{\pi \cancel{\text{rad}}}\right)$$

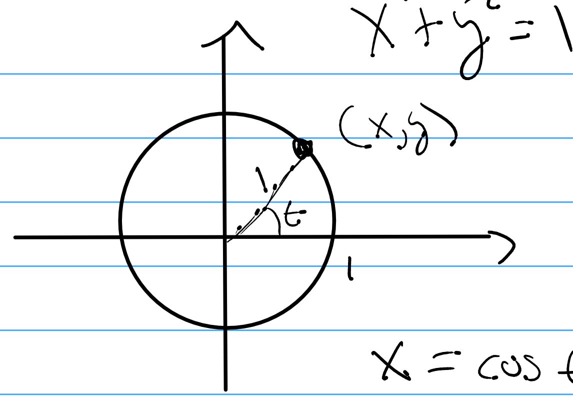
$$\pi = 180^\circ$$

Ch6 6.7

2

trig





$x = \cos t$

$y = \sin t$

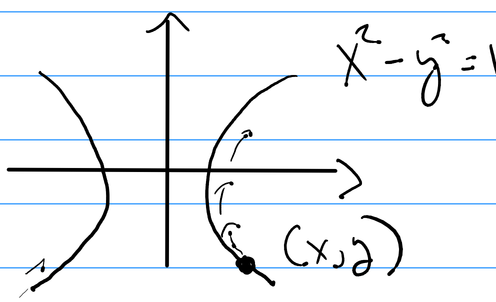
$\sin\theta = \frac{opp}{hyp}$

$\cos\theta = \frac{adj}{hyp}$

$\vdots$

$\cos^2\theta + \sin^2\theta = 1$

6.7 hyperbolic functions



$x = \cosh t$

$y = \sinh t$

Def

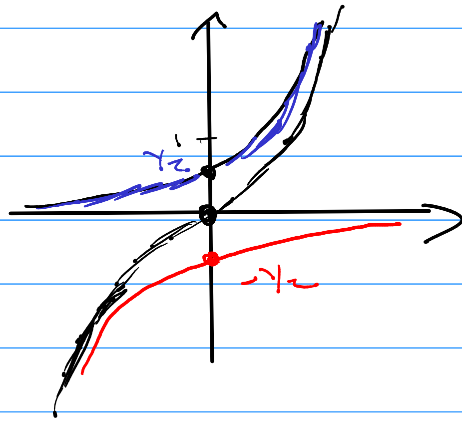
$\sinh x = \dfrac{e^x - e^{-x}}{2}$ $\qquad$ $\operatorname{csch} x = \dfrac{1}{\sinh x}$

$\cosh x = \dfrac{e^x + e^{-x}}{2}$ $\qquad$ $\operatorname{sech} x = \dfrac{1}{\cosh x}$

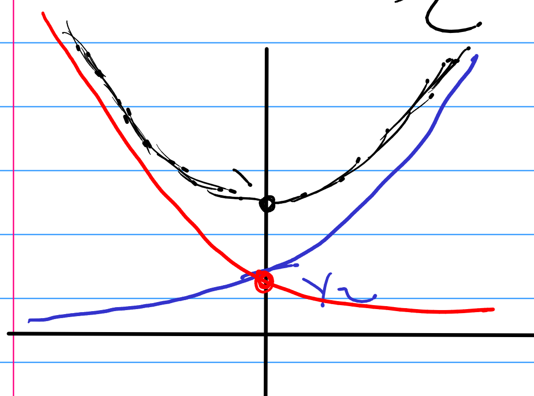
$\tanh x = \dfrac{\sinh x}{\cosh x}$ $\qquad$ $\coth x = \dfrac{\cosh x}{\sinh x}$

Graphs

$$\sinh x = \frac{e^{x} - e^{-x}}{2} = \underline{\tfrac{1}{2}e^{x}} + \underline{-\tfrac{1}{2}e^{-x}}$$

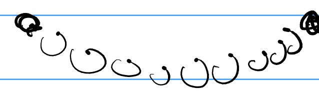


$$\cosh x = \frac{e^{x} + e^{-x}}{2} = \tfrac{1}{2}e^{x} + \tfrac{1}{2}e^{-x}$$



caternary

$$y = c + a\cosh(x/a)$$



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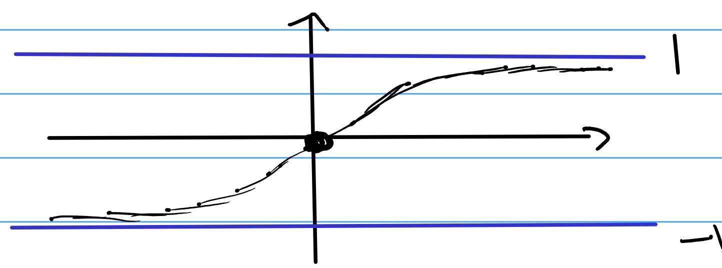
tanh x $= \dfrac{\sinh x}{\cosh x}$



Identities:

$$\boxed{\cosh^2 x - \sinh^2 x = 1}$$

show $\left(\dfrac{e^{x}+e^{-x}}{2}\right)^2 - \left(\dfrac{e^{x}-e^{-x}}{2}\right)^2$

$$= \frac{1}{4}\left[\left(\cancel{e^{2x}} + 2 + \cancel{e^{-2x}}\right) - \left(\cancel{e^{2x}} - 2 + \cancel{e^{-2x}}\right)\right]$$

$$= \frac{1}{4} 4 = 1$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

Even, Odd.

Even $\quad \cosh(-x) = \cosh(x)$

Odd $\quad \sinh(-x) = -\sinh(x)$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

$\boxed{\text{Derivatives}}$

$$\frac{d}{dx}[\sinh x] = \frac{d}{dx}\left[\frac{e^x - e^{-x}}{2}\right] = \frac{1}{2}\frac{d}{dx}[e^x - e^{-x}]$$

$$= \frac{1}{2}[e^x + e^{-x}] = \frac{e^x + e^{-x}}{2} = \cosh x$$

① $(\sinh x)' = \cosh x$

② $(\cosh x)' = \left(\frac{e^x + e^{-x}}{2}\right)' = \frac{e^x - e^{-x}}{2} =$

$$= \sinh x$$

(3) $(\tanh x)' = \left(\frac{\sinh x}{\cosh x}\right)' = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$

$= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$

(4) $(\coth x)' = -\operatorname{csch}^2 x$

(5) $(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x$

(6) $(\operatorname{csch} x)' = -\operatorname{csch} x \coth x$

Inverse

$y = \sinh x \overset{\text{invert}}{\Longleftrightarrow} \boxed{x = \sinh y}$

Algebra?

$\boxed{y = \sinh^{-1}(x)}$

if $x = \sinh y$

$x = \frac{e^y - e^{-y}}{2}$

$2x = e^y - \frac{1}{e^y} \to 2xe^y = e^{2y} - 1$

$\to e^{2y} - 2xe^y - 1 = 0$

$\to (e^y)^2 - 2x(e^y) - 1 = 0$

$e^y = \frac{2x + \sqrt{4x^2 + 4}}{2} = x + \sqrt{x^2+1}$

$$e^y = x + \sqrt{x^2+1}$$

$$y = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\boxed{y = \sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)}$$

Do same for ...

$$y = \cosh(x) \xleftrightarrow{\text{invert}} x = \cosh(y)$$

$x \geq 0$ $\qquad$ $y \geq 0$

$y \geq 1$ $\qquad$ $x \geq 1$

Algebra

$$y = \cosh^{-1}(x)$$

$$y = \cosh^{-1}(x) = \ln\left(x + \sqrt{x^2-1}\right) \quad x \geq 1$$

and $\quad y = \tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

$$-1 < x < 1$$

More Derivatives

$$\frac{d}{dx}\left[\sinh^{-1}(x)\right] = ?$$

$$\boxed{y = \sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)}$$

tech #1

$$\frac{d}{dx}\left[\sinh^{-1}(x)\right] = \frac{d}{dx}\left[\ln\left(x + (x^2+1)^{1/2}\right)\right]$$

$$= \frac{1}{x + \sqrt{x^2+1}} \cdot \left(1 + \frac{1}{\cancel{2}}(x^2+1)^{-1/2} \cdot \cancel{2}x\right)$$

$$= \frac{1}{\cancel{\left(x + \sqrt{x^2+1}\right)}} \cdot \left(\frac{\cancel{\sqrt{x^2+1} + x}}{\sqrt{x^2+1}}\right)$$

$$\boxed{\left(\sinh^{-1} x\right)' = \frac{1}{\sqrt{x^2+1}}}$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \sinh^{-1} x + C$$

tech #2

$$y = \sinh x \xleftrightarrow{\text{invert}} \boxed{x = \sinh(y)}$$

$$\Big\} \text{ Algebra}$$

$$\boxed{y = \sinh^{-1}(x)}$$

or Implicit Derivative

$$1 = \cosh(y) \cdot y'$$

So $y' = \dfrac{1}{\cosh(y)}$

Know $\sinh(y) = x$

$\cosh^2 t - \sinh^2 t = 1$

$\Downarrow$

$\cosh t = \sqrt{1 + \sinh^2 t}$

$$\boxed{y' = \frac{1}{\sqrt{1+x^2}}}$$

"Do" for the rest...

$(\cosh^{-1} x)' = \dfrac{1}{\sqrt{x^2-1}}$ $\qquad$ $(\operatorname{sech}^{-1} x)' = \dfrac{-1}{x\sqrt{1-x^2}}$

$(\tanh^{-1} x)' = \dfrac{1}{1-x^2}$ $\qquad$ $(\coth^{-1} x)' = \dfrac{1}{1-x^2}$

$(\operatorname{csch}^{-1} x)' = \dfrac{-1}{|x|\sqrt{x^2+1}}$

and $\rightarrow$ Integrals $\qquad \displaystyle\int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1} x + C$

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examples

$$y = \left[\sinh\left(x^2 + 3\sin x - \frac{\ln x}{\tan x}\right)\right]^3$$

$$y' = 3\left[\sinh\left(x^2 + 3\sin x - \frac{\ln x}{\tan x}\right)\right]^2 \cdot \left[\text{deriv of inside}\right]$$

$$y' = 3\left[\sinh\left(x^2+3\sin x - \frac{\ln x}{\tan x}\right)\right]^2$$

$$\cdot \cosh\left(x^2+3\sin x - \frac{\ln x}{\tan x}\right)$$

$$\cdot \left[2x + 3\cos x - \left(\frac{\frac{1}{x}\tan x - \ln x \sec^2 x}{\tan^2 x}\right)\right]$$

Integrals ① $\int (\quad) dx$

recognize this as an obvious derivative

$$\int \operatorname{sech}^2 x \, dx = \tanh x + C$$

② Substitution

$$\int f(g(x))\, g'(x)\, dx \Rightarrow \int f(u)\, du$$

let $u = g(x)$

$du = g'(x)dx$

recognize an obvious derivative

Integrals

$$\int \sinh x (\cosh x)^2 \, dx = \int u^2 \, du$$

let $u = \cosh x$

$du = \sinh x \, dx$

$$= \frac{1}{3} u^3 + C$$

$$= \boxed{\frac{1}{3} \cosh^3 x + C}$$

$$\int (\sec x)^2 (\tan x)^3 \, dx = \int u^3 \, du$$

let $u = \tan x$

$du = \sec^2 x \, dx$

$$= \frac{1}{4} u^4 + C$$

$$= \boxed{\frac{1}{4} \tan^4 x + C}$$

$$\int \frac{\sinh(\sqrt{x})}{\sqrt{x}} \, dx = 2 \int \sinh u \, du$$

let $u = \sqrt{x}$

$2 du = \frac{1}{\sqrt{x}} dx$

$$= 2 \cosh u + C$$

$$= \boxed{2 \cosh(\sqrt{x}) + C}$$

$$\int \frac{\sinh(\sqrt{x})}{\sqrt{x}} \, dx = \int \frac{\sinh(u)}{\cancel{u}} 2\cancel{u} \, du$$

let $u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}} dx \rightarrow dx = 2\sqrt{x} \, du = 2u \, du$

$$= 2 \int \sinh(u) \, du = \boxed{2 \cosh \sqrt{x} + C}$$

$$\frac{-1}{|x|\sqrt{x^2+1}} = (\operatorname{csch}^{-1}x)'$$

ex: $$\boxed{\int \frac{1}{\sqrt{x^4+4x^2}}\,dx} = \int \frac{1}{\sqrt{4x^2\left(\frac{x^2}{4}+1\right)}}\,dx$$

$$= \overbrace{\frac{1}{2}\int \frac{1}{|x|\sqrt{\left(\frac{x}{2}\right)^2+1}}\,dx}^{du} = \int \frac{1}{|2u|\sqrt{u^2+1}}\,du$$

let $u = x/2$

$du = \frac{1}{2}dx$

$$= \frac{1}{2}\int \frac{1}{|u|\sqrt{u^2+1}}\,du$$

$$= \boxed{-\frac{1}{2}\operatorname{csch}^{-1}(x/2) + C}$$

6.8 limits of indeterminant forms

$$\lim_{x\to 1} x^2+2 = \boxed{3}$$

$$\lim_{x\to 1} \frac{1}{|x-1|} = \frac{1}{0} = \boxed{+\infty}$$

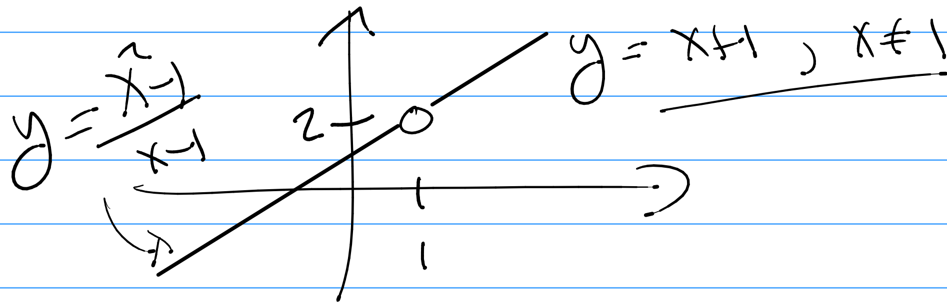
goes to zero (always pos)

$$\lim_{x\to 1} \frac{x-1}{x+2} = \frac{0}{3} = \boxed{0}$$

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$$

$$=\lim_{x\to 1}(x+1)=\boxed{2}$$

$$\boxed{\frac{x^2-1}{x-1}}=\frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}=\boxed{x+1,\ x\neq 1}$$



$$\lim_{x\to 1}\frac{\sin(x-1)}{\ln(x)}=\frac{?}{0}$$

type $\frac{0}{0}$ limit

Indeterminant forms.

type $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty-\infty$, 0^0, ∞^0, 1^∞

$\boxed{\text{L'Hospital's Rule}}$ type $\frac{0}{0}$ or $\frac{\infty}{\infty}$

means: $\frac{f(x)}{g(x)}$ and ① $f \to 0$ and $g \to 0$

or

② $f \to \pm\infty$ and $g \to \pm\infty$

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$ if the second limit exists

$\boxed{\text{So}}$ $$\lim_{x\to 1} \frac{\sin(x-1)}{\ln(x)} = \lim_{x\to 1} \frac{\cos(x-1)}{\frac{1}{x}} = \lim_{x\to 1} x\cos(x-1)$$

type $\frac{0}{0}$ $$= 1 \cdot \cos(1-1) = 1 \cdot 0 = \boxed{0}$$

$\boxed{\text{So}}$ $$\lim_{x\to 1} \frac{\sin(x-1)}{x^2} = \frac{0}{1} = \boxed{0}$$

$\boxed{\text{So}}$ $$\lim_{x\to 0} \frac{3x^2+2x}{2x^2-x} = \lim_{x\to 0} \frac{\cancel{x^2}(3+2/x)}{\cancel{x^2}(2-1/x)}$$

$$= \lim_{x\to 0} \frac{3+2/x}{2-1/x} = \frac{?}{0}$$

$\boxed{\text{L'H}}$ $$\lim_{x\to 0} \frac{6x+2}{4x-1} = \frac{2}{-1} = \boxed{-2}$$

$$\lim_{x\to\infty}\frac{3x^2+2x}{2x^2-x} = \lim_{x\to\infty}\frac{\cancel{x^2}(3+2/x)}{\cancel{x^2}(2-1/x)}$$

$$= \lim_{x\to\infty}\frac{3+\cancel{2/x}^{\to 0}}{2-\cancel{1/x}^{\to 0}} = \boxed{\frac{3}{2}}$$

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by L'H $\displaystyle\lim_{x\to\infty}\frac{6x+2}{4x-1} = \lim_{x\to\infty}\frac{6}{4} = \boxed{\frac{3}{2}}$

type $\frac{\infty}{\infty}$

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type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ use L'Hospital (if needed)

$\boxed{\infty-\infty}$?

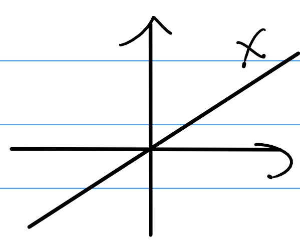
$$\lim_{x\to\pi/2^-}(\sec x - \tan x) = \lim_{x\to\pi/2^-}\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$$

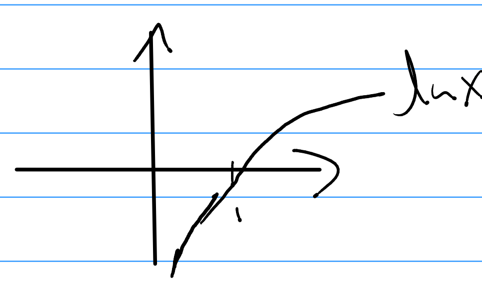
$$= \lim_{x\to\pi/2^-}\frac{1-\sin x}{\cos x} \overset{\boxed{L'H}}{=} \lim_{x\to\pi/2^-}\frac{\cos x}{\sin x} = \frac{0}{1} = \boxed{0}$$

type $\frac{0}{0}$

$\boxed{\text{Ex}}$ $\displaystyle\lim_{x\to\infty}\frac{x}{\ln(x)}$

type $\frac{\infty}{\infty}$





$\rightarrow \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \boxed{\infty}$

$\rightarrow \lim_{x\to\infty} \frac{e^x}{x^p} \overset{LH}{=} \lim_{x\to\infty} \frac{e^x}{p\,x^{p-1}} = \lim_{x\to\infty} \frac{e^x}{\underbrace{p(p-1)(p-2)\cdots(1)}_{p!}}$

$p > 1$

type $\frac{\infty}{\infty}$ type $\frac{\infty}{\infty}$

$= \boxed{\infty}$

$\boxed{\text{type } 0 \cdot \infty}$

Use the fact that

if $f \to 0$ $f = \frac{1}{\underbrace{1/f}_{\to\infty}}$

if $f \to \infty$ $f = \frac{1}{\underbrace{1/f}_{\to 0}}$

$\boxed{\text{ex}}$ $\lim_{x\to\infty} \underbrace{x}_{\to\infty} \sin(\underbrace{\pi/x}_{\to 0}) = \lim_{x\to\infty} \frac{\sin(\pi/x)}{1/x}$

type $\infty \cdot 0$ type $\frac{0}{0}$

$\overset{LH}{=} \lim_{x\to\infty} \frac{\cos(\pi/x)\cancel{\left(\frac{-1}{x^2}\right)}}{\cancel{\left(-1/x^2\right)}} = \lim_{x\to\infty} \cos(\pi/x) = \boxed{1}$

type 0^0, ∞^0, 1^∞

$$f^g = e^{\ln f^g} = e^{g\ln(f)}$$

$$\lim f^g = \lim e^{g\ln(f)} = e^{\lim[g\ln(f)]}$$

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ex $\lim_{x\to 0^+} x^{\sqrt{x}}$ $\quad x^{\sqrt{x}} = e^{\boxed{\sqrt{x}\ln x}}$

type 0^0

$$\lim_{x\to 0^+} x^{\sqrt{x}} = e^{\lim_{x\to 0^+}\sqrt{x}\ln x} = e^{\lim_{x\to 0^+}\frac{\ln x}{x^{-1/2}}}$$

$$\underset{LH}{=} e^{\lim_{x\to 0^+}\frac{1/x}{-1/2\, x^{-3/2}}} = e^{\lim_{x\to 0^+} -2x^{1/2}} = e^0$$

$$= \boxed{1}$$
