Math 243

Homework $\longrightarrow$ WebAssign

$$\sqrt{x} = x^{1/2}$$

$$y = \frac{4x}{5 - \sqrt{x}} \qquad y' = \frac{(4x)'(5-\sqrt{x}) - (4x)(5-\sqrt{x})'}{(5-\sqrt{x})^2}$$

$$y' = \frac{(4)(5-\sqrt{x}) - (4x)\left(-\frac{1}{2}x^{-1/2}\right)}{(5-\sqrt{x})^2}$$

$$= \frac{20 - 4x^{1/2} + 2x^{1/2}}{(5-x^{1/2})^2} = \frac{20 - 2x^{1/2}}{(5-x^{1/2})^2} = \boxed{\frac{2(10-x^{1/2})}{(5-x^{1/2})^2}}$$

$$\boxed{y' = \frac{2(10-\sqrt{x})}{(5-\sqrt{x})^2}}$$

$y' = 0$

y' dne

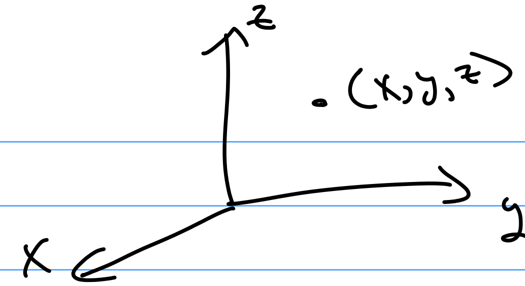
$$y = \frac{\sqrt[3]{t}}{t-1} = \frac{t^{1/3}}{t-1}$$

$$y' = \frac{(t^{1/3})'(t-1) - (t^{1/3})(t-1)'}{(t-1)^2}$$

$$y' = \frac{\left(\frac{1}{3}t^{-2/3}\right)(t-1) - (t^{1/3})(1)}{(t-1)^2}$$

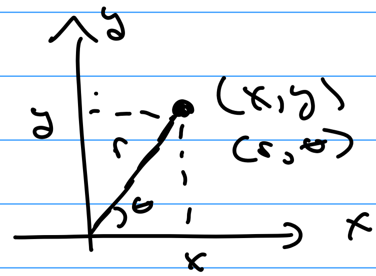
$y' =$ Finish!

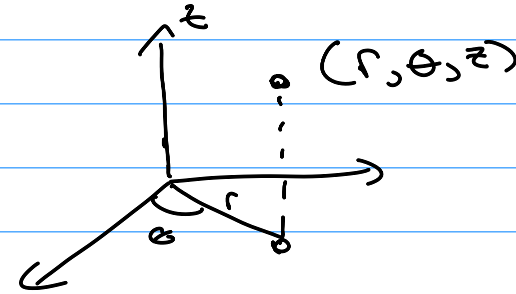
12.1 $\mathbb{R}^3$ (3D space)

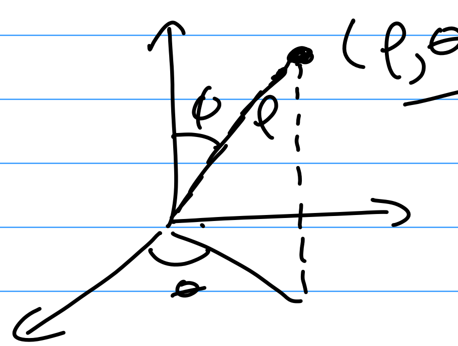


→ cartesian coord.

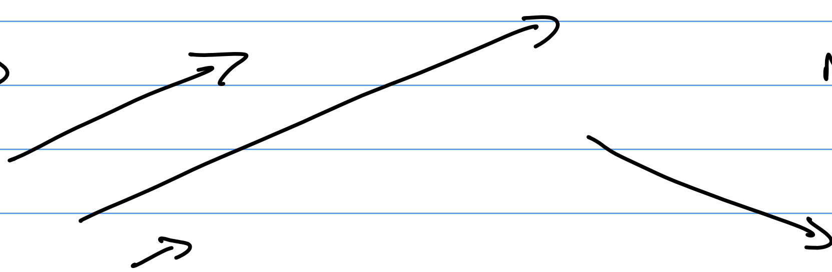
2D







12.2



New object that has
① magnitude
② direction

∴ $\boxed{\text{Vector}}$

toy: Vector rules: ?

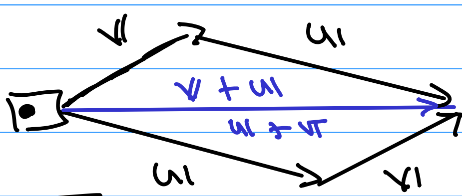
Notation: bold font lower case

or double font

$\mathbb{v}$

or $\vec{v}$

① Sum $\mathbb{v} + \mathbb{u}$



③ $\boxed{c\,\mathbb{v}} = \underbrace{\mathbb{v} + \mathbb{v} + \mathbb{v} + \cdots + \mathbb{v}}_{c \text{ occur.}}$ Scalar mult.

$c\vec{v}$ a) New mag is $|c| \cdot$ length of $\vec{v}$

b) New direction
- $c > 0$ same direction
- $c < 0$ opposite direction

Note: $\vec{0}$ is the vector of no mag. and no direction

$0 \cdot \vec{v} = \vec{0}$

$c \cdot \vec{0} = \vec{0}$

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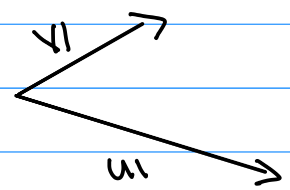
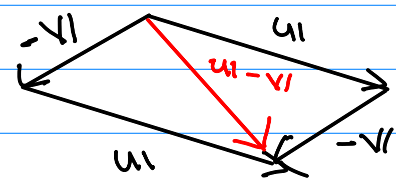
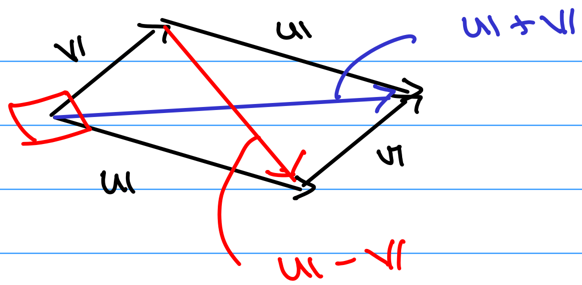
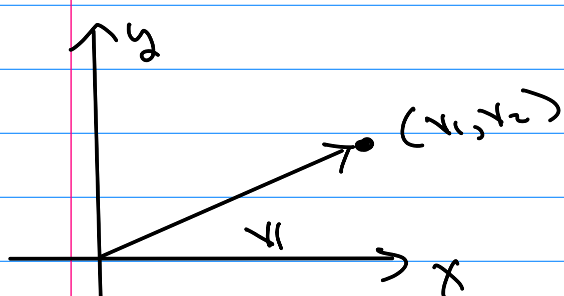
③ Difference $\vec{u} - \vec{v} = \vec{u} + (-1)\vec{v}$





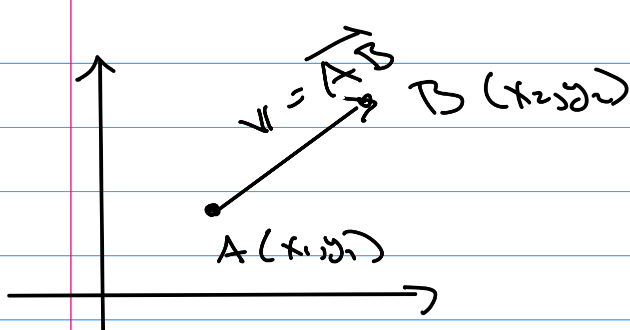
So



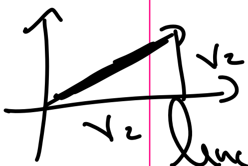


2D $\vec{v} = \langle v_1, v_2 \rangle$

3D $\vec{v} = \langle v_1, v_2, v_3 \rangle$



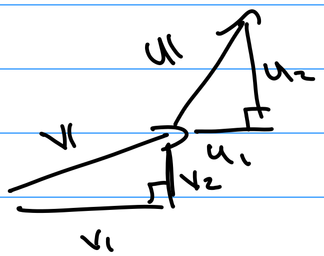
$\vec{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$ (2D)



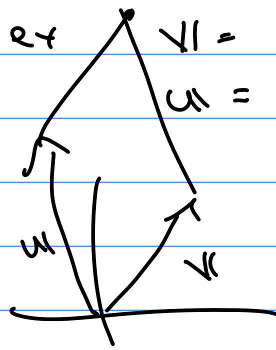
length of $\mathbf{v}$ 2D $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$

3D $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

Ops ① $\mathbf{v} + \mathbf{u} = \langle v_1 + u_1, v_2 + u_2 \rangle$



ex



$\mathbf{v} = \langle 3, 5 \rangle$

$\mathbf{u} = \langle -2, 7 \rangle$

$\mathbf{v} + \mathbf{u} = \langle 1, 12 \rangle$

② $\mathbf{v} - \mathbf{u} = \langle v_1 - u_1, v_2 - u_2 \rangle$

③ $c\mathbf{v} = \langle cv_1, cv_2 \rangle$

Properties:

① $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

② $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$

③ $\mathbf{a} + \mathbf{0} = \mathbf{a}$ ($\mathbf{0}$ is the vector sum identity)

④ $\mathbf{a} + (-1)\mathbf{a} = \mathbf{0}$ ($-\mathbf{a}$ is the vector sum inverse)

⑤ $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$

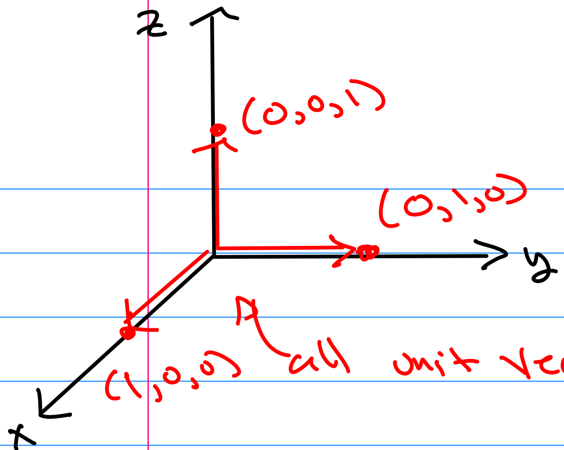
⑥ $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$

⑦ $(cd)\mathbf{a} = c(d\mathbf{a})$

⑧ $1 \cdot \mathbf{a} = \mathbf{a}$

So far on Vectors

or $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$



all unit vectors (length = 1)

Standard basis

$\mathbf{i} = \langle 1, 0, 0 \rangle$

$\mathbf{j} = \langle 0, 1, 0 \rangle$

$\mathbf{k} = \langle 0, 0, 1 \rangle$

Why?

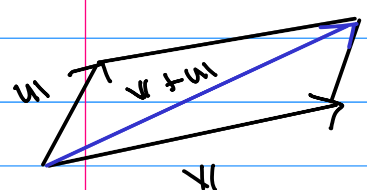
$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle$$

$$\boxed{\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}}$$

b/c $\mathbf{v} = v_1 \langle 1, 0, 0 \rangle + v_2 \langle 0, 1, 0 \rangle + v_3 \langle 0, 0, 1 \rangle$

$= \langle v_1, 0, 0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0, 0, v_3 \rangle$

$= \langle v_1, v_2, v_3 \rangle$



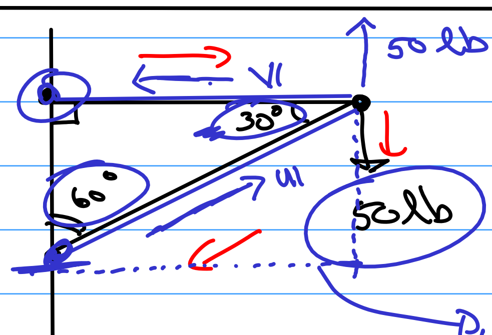
$\mathbf{u} = \langle 1, 2 \rangle \qquad \mathbf{v} = \langle 4, 1 \rangle$

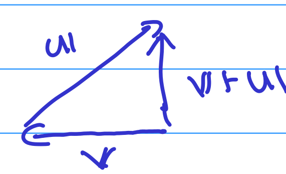
$\mathbf{u} + \mathbf{v} = \langle 5, 3 \rangle$

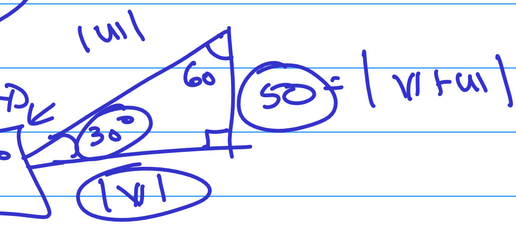
$\mathbf{u} = 1 \cdot \mathbf{i} + 2\mathbf{j} \qquad \mathbf{v} = 4\mathbf{i} + 1 \cdot \mathbf{j}$

$$\mathbf{u} + \mathbf{v} = (\mathbf{i} + 2\mathbf{j}) + (4\mathbf{i} + \mathbf{j}) = \boxed{5\mathbf{i} + 3\mathbf{j}}$$

Word problems



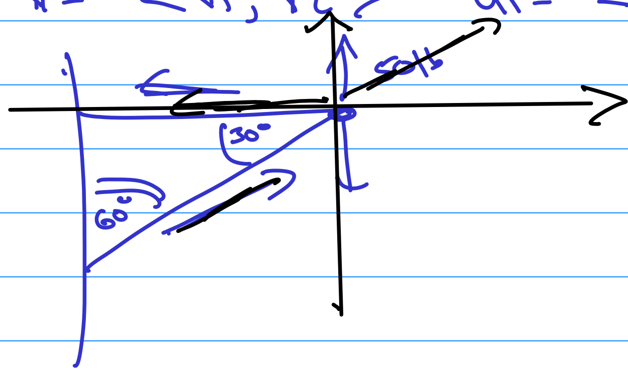


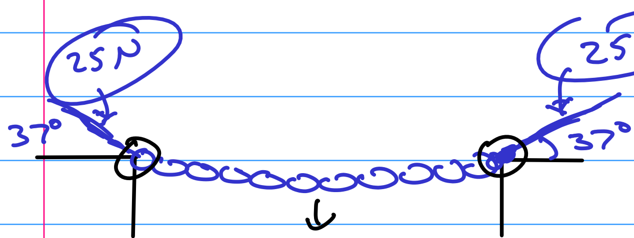


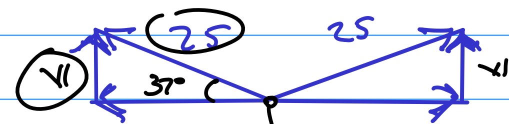
$\sin 30° = \dfrac{50}{|\mathbf{u}|} \rightarrow |\mathbf{u}| = \boxed{50 \sin 30°}$

$\tan 30° = \dfrac{50}{|\mathbf{v}|} \rightarrow |\mathbf{v}| = \boxed{50 \tan 30°}$

$\mathbf{v} = \langle v_1, v_2 \rangle \quad \mathbf{u} = \langle u_1, u_2 \rangle$





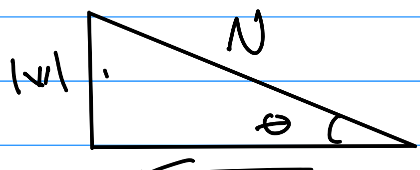


gravity = weight of chain

$2\mathbf{v}$ = weight of chain

$\sin 37° = \dfrac{|\mathbf{v}|}{25}$

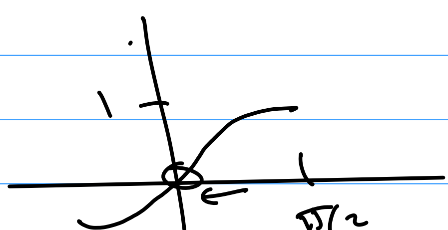
$25 \sin 37° = |\mathbf{v}|$ weight $= 50 \sin 37°$



$2N \sin\theta$ = weight of chain = $\boxed{\text{const.}}$ = K

$2N \sin\theta = K$

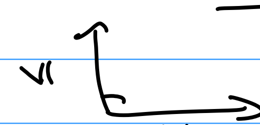
$\boxed{N} = \dfrac{K}{2\sin\theta}$ as $\theta \to 0$



12.3 / 12.4 Two types of "multiply"

$\boxed{12.3}$ Dot Product take two vectors → return a scalar

contribution



Def $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

ex $\mathbf{a} = \langle 1,-1,3\rangle \qquad \mathbf{b} = \langle 0,4,2\rangle$

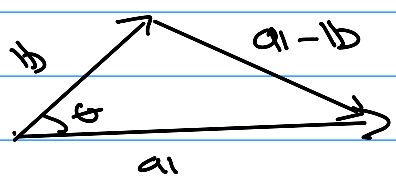
$$\mathbf{a}\cdot\mathbf{b} = 0 - 4 + 6 = 2$$

$$\mathbf{i}\cdot\mathbf{j} = \langle 1,0,0\rangle\cdot\langle 0,1,0\rangle = 0$$

$$\langle 3,0,0\rangle\cdot\langle 2,0,0\rangle = 6$$
$$3\mathbf{i} \cdot 2\cdot\mathbf{i}$$

Properties

1. $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$
3. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$
4. $(c\,\mathbf{a})\cdot\mathbf{b} = c(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(c\mathbf{b})$
5. $\mathbf{0}\cdot\mathbf{a} = 0$



law of cosines

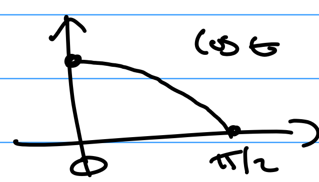
$$\underline{|\mathbf{a}-\mathbf{b}|^2} = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta$$

$\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2 \qquad (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = \text{right}$

$$\mathbf{a}\cdot\mathbf{a} - 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b} = \text{right}$$

$$\Rightarrow \cancel{|a|^2} + \cancel{2}\mathbf{a}\cdot\mathbf{b} + \cancel{|\mathbf{b}|^2} = \cancel{|a|^2} + \cancel{|\mathbf{b}|^2} + \cancel{2}|a||b|\cos\theta$$

$$\boxed{\mathbf{a}\cdot\mathbf{b} = |a||b|\cos\theta}$$

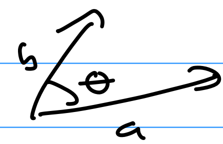


So $\boxed{a \perp b \text{ if and only if } a \cdot b = 0}$

Know: $a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$

or

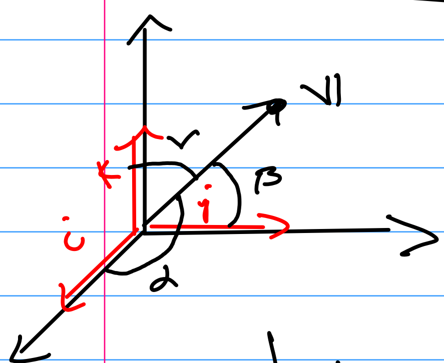
$a \cdot b = |a|\,|b| \cos\theta$



4^{th} representation of vectors..

① $\vec{v}$ ② $v = \langle v_1, v_2, v_3 \rangle$ ③ $v = v_1 i + v_2 j + v_3 k$

④ direction angles $\quad \cos\theta = \dfrac{a \cdot b}{|a|\,|b|}$



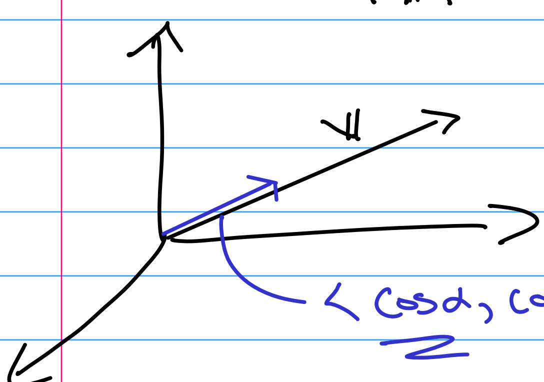
So $\cos\alpha = \dfrac{v_1}{|v|}$, $\cos\beta = \dfrac{v_2}{|v|}$

$\cos\gamma = \dfrac{v_3}{|v|}$

$$\left| \langle \cos\alpha, \cos\beta, \cos\gamma \rangle \right| = 1$$

So ① $v = |v| \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$

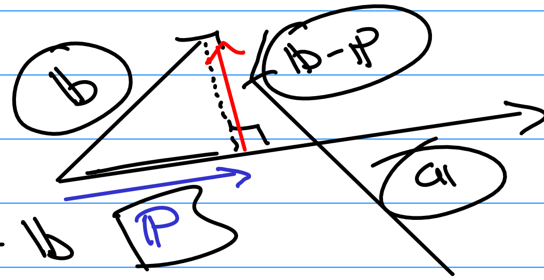
② $\dfrac{v}{|v|} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$



$\langle \cos\alpha, \cos\beta, \cos\gamma \rangle$ unit vector in direction of v

another application of $\mathbf{a}\cdot\mathbf{b}$

Projections

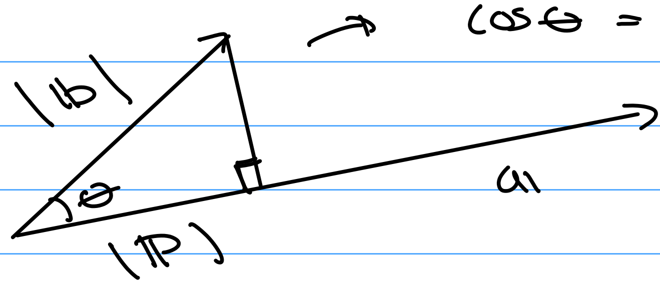


$\mathbf{p}$ = vector projection of $\mathbf{b}$ onto $\mathbf{a}$

$|\mathbf{p}|$ = scalar projection of $\mathbf{b}$ onto $\mathbf{a}$

textbook notation $\quad \mathbf{p} = \mathrm{proj}_{\mathbf{a}}(\mathbf{b})$

$$|\mathbf{p}| = \mathrm{comp}_{\mathbf{a}}(\mathbf{b})$$



$\rightarrow \quad \cos\theta = \dfrac{|\mathbf{p}|}{|\mathbf{b}|} \rightarrow |\mathbf{p}| = |\mathbf{b}|\,\boxed{\cos\theta}$

but know $\quad \boxed{\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \qquad \boxed{\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}} = \cos\theta}$

$$|\mathbf{p}| = \cancel{|\mathbf{b}|}\cdot\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\cancel{|\mathbf{b}|}}$$

Scalar proj of $\mathbf{b}$ onto $\mathbf{a}$ $= |\mathbf{p}| = \boxed{\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}}$ scalar proj.

Vector proj of $\mathbf{b}$ onto $\mathbf{a}$ $= \mathbf{p} = |\mathbf{p}|\cdot\dfrac{\mathbf{a}}{|\mathbf{a}|}$

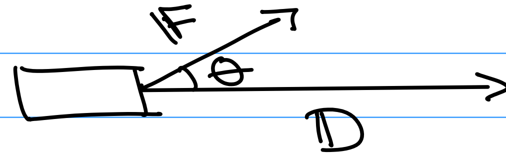
so $\quad \mathbf{p} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a} = \boxed{\dfrac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\cdot\mathbf{a}}$ Vector proj.

Work ← scalar quantity

$$W = \mathbf{F} \cdot \mathbf{D}$$

or

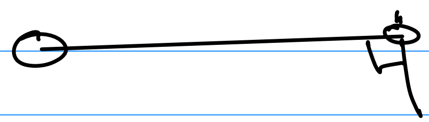
$$W = |\mathbf{F}||\mathbf{D}| \cos\theta$$

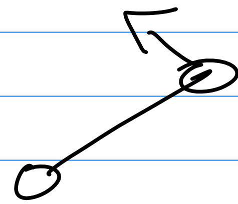


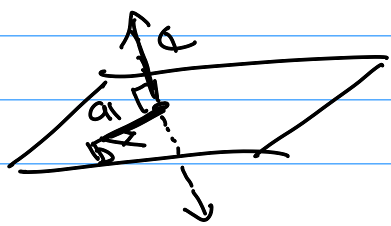
12.4 Cross Product

take two vectors in 3D and return a vector that is $\perp$ to both

and...







$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$

$$\mathbf{a} \cdot \mathbf{c} = 0 \qquad \mathbf{b} \cdot \mathbf{c} = 0$$

$$\begin{cases} a_1 c_1 + a_2 c_2 + a_3 c_3 = 0 \\ b_1 c_1 + b_2 c_2 + b_3 c_3 = 0 \end{cases}$$

Solve:

$$c_1 = a_2 b_3 - a_3 b_2$$
$$c_2 = a_3 b_1 - a_1 b_3$$
$$c_3 = a_1 b_2 - a_2 b_1$$

$$\boxed{\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle}$$

$\mathbf{a} \times \mathbf{b}$ is $\perp$ to both $\mathbf{a}, \mathbf{b}$

Def determinant of a matrix

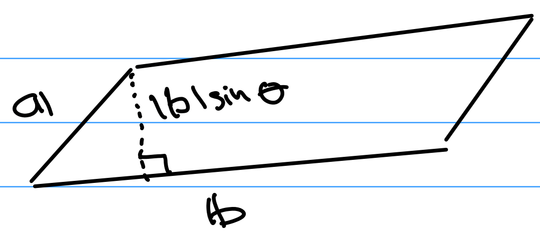
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

3×3 $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

then $\boxed{\mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}$ ✓

and $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$



$|a||b| \sin\theta$ = area of parallelogram