The nodal count mystery

Peter Kuchment, Texas A&M University
Partially supported by NSF

21st Midwest Geometry Conference
Wichita State University, KS
March 11–13 2016
Table of contents

1. Nodal patterns of eigenfunctions
2. Properties of the nodal sets and partitions
3. Nodal count - Courant theorem
4. Minimal partitions and Courant sharp eigenfunctions
5. Critical partitions, Morse index, and nodal deficiency

Peter Kuchment, Texas A&M University Partially supported by The nodal count mystery
Nodal patterns (Chladni figures) observed and studied by Leonardo da Vinci (1452 – 1519), Galileo Galilei (1564–1642), Robert Hooke (1635–1703), Ernst Chladni (1756 – 1827), ...
More nodal patterns
Even more nodal patterns
Even more nodal patterns

Peter Kuchment, Texas A&M University Partially supported by

The nodal count mystery
Mysteries still abound

Studied in: math. physics, number theory, medical imaging (sic!).

• Local structure of nodal sets.
  Smoothness, intersection angles - easy.
  Curvature of joint nodal sets - HARD (Bourgain, Rudnick, Agranovsky & Quinto, Ambartsoumian & P.K., ..., Finch et al)
• How large is the nodal set? Yau’s conjecture: $H_{n-1}(N_\lambda) \sim \lambda^{1/2}$.
  Proven in analytic case by Fefferman and Donnelly. In smooth case - HARD, work is very active (Colding, Hezari, Mangoubi, Minicozzi, I. Polterovich, Rudnick, Sarnak, Sodin, Zelditch).
• How many nodal domains does the $n$th eigenfunction have?
  (Courant, Helffer & T. Hoffman-Ostenhof, Berkolaiko & P.K. & Smilansky, Bourgain, Bogomolny & Schmit, Nazarov & Sodin)
• Inverse problems. Band, Hald, Klawonn, J. McLaughlin, Smilansky. Nadirashvili, K. Uhlenbeck, ...
Nodal count in 1D – Sturm’s theorem

\[ H = -\frac{d^2}{dx^2} + q(x) \]

on \([a, b]\) with Dirichlet boundary conditions \(u(a) = u(b) = 0\).

**Sturm’s Theorem:**

Let \(\lambda_1 \leq \lambda_2 \leq \ldots\) be the spectrum and \(\psi_n(x)\) - eigenfunctions of \(H\). Then \(\psi_n\) changes sign exactly \(n - 1\) times inside \([a, b]\). Thus, there are exactly \(n\) nodal domains, where \(\psi_n\) has constant sign.
Nodal count in higher dimensions

\[ H = -\Delta + q(x) \]

in a bounded domain \( \Omega \subset \mathbb{R}^d \) with Dirichlet boundary conditions \( u|_{\partial \Omega} = 0 \). \((\lambda_1(\Omega), \psi_1(x)), (\lambda_2(\Omega), \psi_2(x)), \ldots\)

**Nodal set** of eigenfunction \( \psi_n \):

\[ \mathcal{Z}_n := \{ x \in \Omega \mid \psi_n(x) = 0 \}. \]

**Nodal domain** of \( \psi_n \): a connected component of \( \Omega \setminus \mathcal{Z}_n \).

**Nodal count** \( \nu_n \): number of nodal domains.

**Courant’s Theorem:**

\[ \nu_n \leq n \]
Courant sharp eigenfunctions are such that $\nu_n = n$. E.g., in 1D, all eigenfunctions are Courant sharp (Sturm). $\lambda_1$ and $\lambda_2$ are always Courant sharp.

Theorem (Pleijel)

For $d > 1$ and large $n$, $\nu_n < \left(\frac{2}{j}\right)^2 n \approx 0.691n$. (improved by Bourgain ’13, also I. Polterovich)

Corollary

For $d > 1$ there are only finitely many Courant sharp eigenfunctions.

Definition $\mu_n := n - \nu_n$ – nodal deficiency.
Two domains for large $n$

Large $n$ and $\nu_n = 2$ (A. Stern, 1925):
Some basic questions are not answered, e.g.

\[ \limsup_{n \to \infty} \nu_n = \infty \]

? ????????????????
Nodal Partitions

Helffer & Hoffman-Ostenhof: Look at possible partitions $P$ of domain $\Omega$:

Q: Which partitions $P = \{\Omega_j\}$ can be nodal partitions of eigenfunctions?
A: Necessarily,
- **Local structure** (smooth, correct intersection angles)
- **Bipartite**:

- **Equipartition**: \( \lambda_1(\Omega_j) = \lambda_n(\Omega) \).
Nodal Partitions - continued

Helffer–Hoffman–Ostenhof functional:

\[ \Lambda(P) := \max_j \lambda_1(\Omega_j) \]


**Minimal** bipartite partitions are exactly nodal partitions of Courant sharp eigenfunctions.

Why? What about the non-Courant-sharp ones?
Critical Partitions


In the billiard case,


*Among all generic equipartitions, the bipartite critical points of \( \Lambda \) are exactly nodal partitions of eigenfunctions.*

*The Morse index is equal to the nodal deficiency \( \mu_n \).*

*In particular, at minimal points, Morse index is zero and thus the eigenfunction is Courant sharp.*

Berkolaiko (2011) (a modification by Colin de Verdiere), in graph case - Morse indices w.r.t. perturbations by magnetic potentials.
Genericity

nodal domain pattern for $-\Delta$

possible nodal domain pattern for $-\Delta + q$
Some references

Thank you very much for the invitation and patience