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Breakthrough in Conformal Mapping

By James Case

Few analytical techniques are better known to students of applied mathematics than conformal mapping. It is the classical method for solving problems in continuum mechanics, electrostatics, and other fields involving the two-dimensional Laplace and Poisson equations. To employ the method, one needs an explicit mapping function from some standard domain—such as the unit disk or upper half plane—to the region of interest. For a broad class of simply connected domains, and a few doubly connected ones, the Schwarz–Christoffel (SC) formula provides the required map. But until quite recently, there was no analogue of the SC formula for multiply connected domains. Today, however, the "connectivity barrier" has been breached in at least two places.

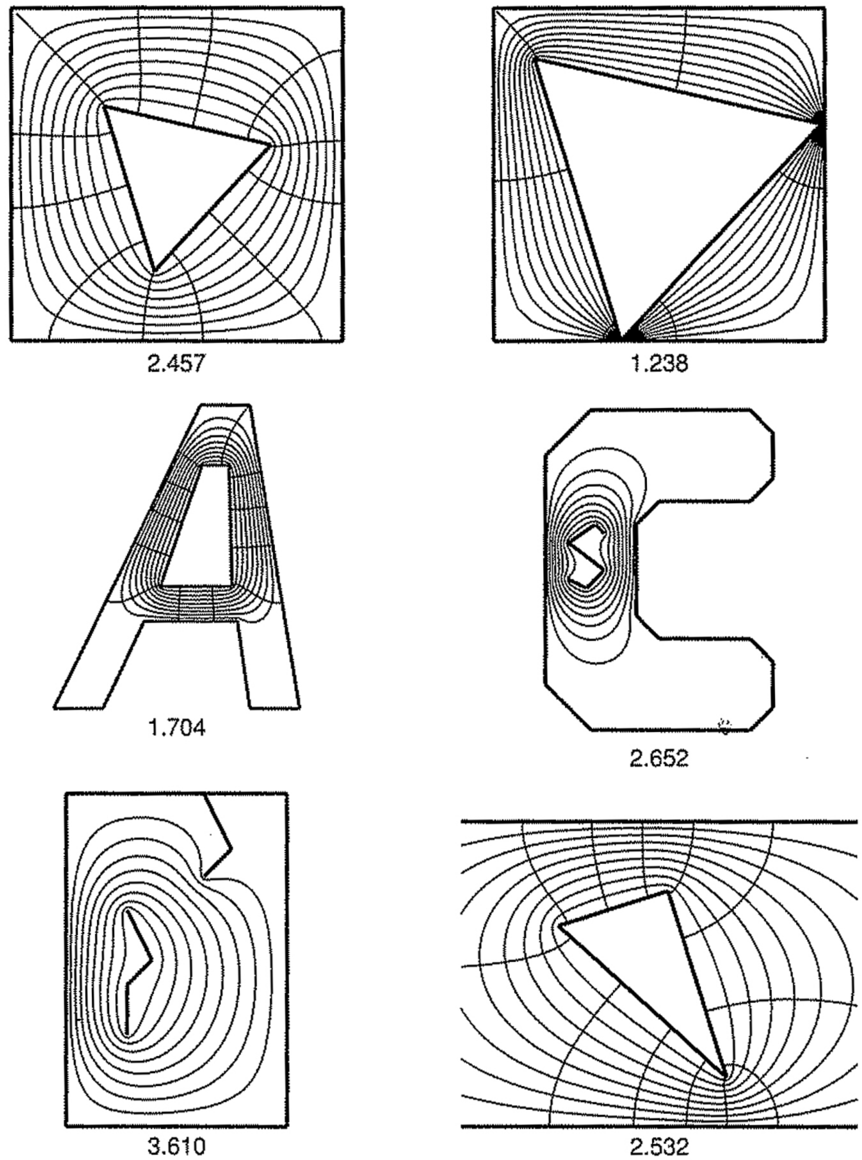

Figure 1. *An annulus can be mapped conformally and univalently onto a doubly connected polygon only if the two share a common Riemann modulus. As a practical matter, it seems necessary to calculate the modulus and the mapping function simultaneously, by successive approximation. Figures 1 and 2 from* Schwarz–Christoffel Mapping *[5].*

Background: Broad Outlines of a Theory of Conformal Mapping

In 1952, Zeev Nehari published what remains the standard treatise on conformal mapping [7]. The seventh and final chapter, on multiply connected domains, begins with a proof that the annulus $\mu < |z| < 1$ can be mapped conformally and univalently onto $\upsilon < |w| < 1$ if and only if $\mu = \upsilon$. Hence, there can be no canonical domain, such as the unit disk or the upper half-plane, to which every doubly connected domain with sufficiently many boundary points is conformally equivalent. Because Riemann was aware that it is possible to map any doubly connected domain in the z-plane conformally and univalently onto an annulus $\upsilon < |w| < 1$, the conformal invariant υ is known as the "Rie-mann modulus" of the domain in question.

To map an annulus $\mu < |z| < 1$ onto a particular doubly connected (polygonal, say) subset of the w-plane, one must first (or simultaneously) identify the Riemann modulus of the target domain. To illustrate what can be done with off-the-shelf numerical mapping software, T.A. Driscoll and L.N. Trefethen offer the examples shown in Figure 1 [5]. The number beneath each map is what they call the "conformal modulus" μ^{-1}.

The situation for multiply connected do-mains is more complex. In this article, D represents a domain of connectivity $n > 2$ in the z-plane. D's "conformal type" is then determined by $3n - 6$ real parameters—which Nehari also described as Riemann moduli—in such a way that D can be mapped conformally onto an image D' of the same connectivity if and only if D and D' agree in all $3n - 6$ Riemann moduli. Although there is no single canonical domain onto which all such D can be mapped conformally and univalently, there do exist several infinite families of "slit domains" such that any D with sufficiently many boundary points can be so mapped onto just one (suitably normalized) member of each family.

Nehari described five such families, three consisting of unbounded domains and the remaining two of subsets of the unit disk. D can be mapped conformally onto (i) the entire unit disk $|w| < 1$ from which $n - 1$ concentric circular slits have been removed, or (ii) an annulus $\upsilon < |w| < 1$ from which $n - 2$ concentric circular slits have been removed. Alternatively, D can be mapped onto the entire w-plane (including the point at infinity) from which (iii) n parallel rectilinear slits, (iv) n rectilinear slits radiating outward from a common center, or (v) n concentric circular slits have been removed. Nehari also explained how the functions mapping D into the various classes (i)–(v) are related to one another.

Finally, let u and v be any two points of D, and let $S(u,v)$ denote the class of functions f on D, analytic and univalent, for which $f(u) = 0$ and $f(v) = \infty$.* As is true for simply connected domains, $S(u,v)$ constitutes a "normal family," so that every continuous functional φ defined on $S(u,v)$ actually attains its maximum and minimum for at least one function in the class. In particular, the functions $f \in S(u,v)$ that furnish the maximum and minimum values of the functional $\psi(f) = |f'(u)|$ map D conformally onto an unbounded concentric circular slit domain and an unbounded radial slit domain, respectively. As in the doubly connected case, the Riemann moduli of D determine which slit domain in each class is the conformal image of D.

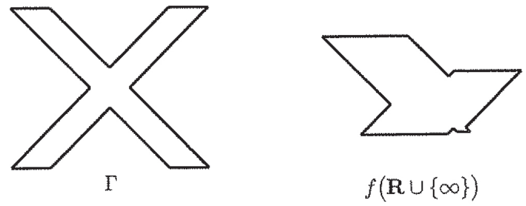

Figure 2. *Inaccurate estimates of the pre-images of the several vertices can frustrate efforts to find the SC formula that maps the unit circle onto a given polygon.*

As of about 1950, then, the broad outlines of a theory of conformal mapping of multiply connected domains were in hand. Only the means of mapping a given D onto an appropriate D' in (i)–(v) were missing. And there the field pretty well languished for the better part of fifty years. Even the intense development that SC mapping techniques have undergone since the late 1970s—when interactive computing became widespread—had failed to alter the status quo.

Breakthrough

Quite recently, the situation began to change dramatically. In a paper published in 2004 in the J*ournal d'Analyse Mathéma-tique*, Tom DeLillo, Alan Elcrat, and John Pfaltzgraff derived a Schwartz–Christoffel formula mapping the exterior of a finite collection of non-intersecting disks onto the exterior of a like number of disjoint polygons. In a session in Sydney, at ICIAM '03, listening to Elcrat speak on his then unpublished work with Pfaltzgraff and DeLillo, Darren Crowdy was led to suspect that a more abstract approach to the questions at issue might lead to additional results. His first paper [1] on the subject, containing an SC formula for mapping the interior of the unit disk with m circular holes onto the interior of a bounded polygon with m polygonal holes, appeared in 2005. A subsequent publication [2] extended his results to unbounded domains.

The phrase "an SC formula" requires explanation. Christoffel (in 1867) and Schwarz (in 1869) published versions of the mapping formula that now bears both their names. Perhaps the most familiar version maps the upper half of the z-plane onto a user-specified polygonal subset of the w-plane. The characteristic feature of all SC mapping functions f is that their derivatives f' can be expressed as products $\Pi_k f_k$ of simpler "canonical functions" f_k. In fact, according to Driscoll and Trefethen, almost every known conformal mapping is an SC map in the foregoing sense, possibly disguised by a prior change of variables.

The geometrical significance of the product form of an SC mapping function is that arg $f' = \Sigma_k \arg f_k$. Thus, for instance, the product of functions of the form $f_k = (z - z_k)^{(-\beta_k)}$ is the derivative of a function f mapping the upper half of the z-plane onto a closed simply connected polyhedral subset of the w-plane with interior angles $\alpha_k\pi = (1 - \beta_k)\pi$ at each of the vertices $w_k = f(z_k)$. The mapping from the unit circle to the same polygon can then be obtained from the Möbius transformation that maps the unit circle onto the upper half plane. To close the boundary polygon, the multipliers β_k must of course add up to 2. To construct such a map, it is necessary to choose the "pre-vertices" z_k with some care, as shown in Figure 2. The map from the annulus $\mu < |z| < 1$ to a polyhedral region with a single polyhedral hole involves canonical functions of the form

$$\Theta(z,\mu) = \Pi_j \, (1 - \mu^{2j-1}z)(1 - \mu^{2j-1}z^{-1}), \tag{1}$$

in which j runs from 1 to ∞. This doubly periodic complex-valued function is closely related to the classical Jacobi elliptic theta functions, and is in fact the Schottky–Klein prime function associated with the annulus $\mu < |z| < 1$.

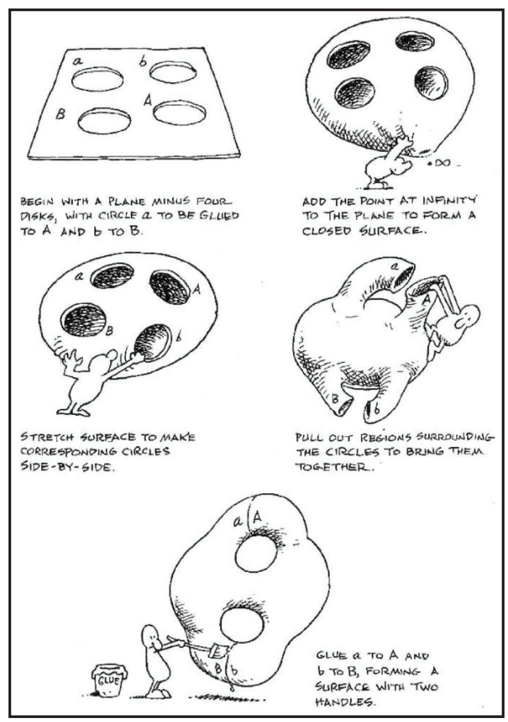

Figure 3. *Constructing the Riemann surface of a planar figure with 2g holes. From* Indra's Pearls *[6].*

Crowdy's 2005 derivation uses properties of the so-called "Schottky group" of Möbius transformations, along with the "Schottky–Klein prime function" associated with any compact Riemann surface. In a minisymposium at the 2006 SIAM Annual Meeting, Crowdy explained prime functions for the benefit of those in the audience (including this reporter) whose ignorance of them was complete. To that end, he illustrated the steps by which a plane from which $2g$ circular holes have been removed can be deformed into a compact Riemann surface of genus g [6] (see Figure 3). In the absence of holes, the construction would yield the familiar Riemann sphere. With two holes, it yields a torus. With $2g$ holes, it yields a sphere with g "handles."

For the non-intersecting circles C_i, $i = 1, \ldots, m$, that lie within the unit circle U: $|z| < 1$, the reflections C_i', $i = 1, \ldots, m$, through U will lie without. By following the steps indicated in Figure 3, it is possible to construct the compact Riemann surface of genus m that corresponds in a natural way to the unit circle from which the interiors of the circles C_i, $i = 1, \ldots, m$, have been removed. The Möbius transformations $\theta_i(z)$ that carry the interiors of the circles

*Actually Nehari asks that members of $S(u,v)$ have a pole of unit order and unit residue at v.

C_i' onto the interiors of their pre-images C_i in U are the generators of the so-called Schottky group of transformations on the Riemann surface.

Given a compact Riemann surface, and an arbitrarily chosen point ζ of that surface, there exists a unique (typically transcendental) function $\omega(z,\zeta)$ with the following properties:

- ω has a simple zero at $z = \zeta$;
- ω is holomorphic in z everywhere on the Riemann surface; and
- ω possesses certain transformation properties under the action of the Schottky group of Möbius transformations.

The prime function corresponding to the Riemann sphere is just $\omega(z,\zeta) = z - \zeta$. Any meromorphic function on the Riemann sphere, such as a polynomial or rational function, can be factored into a product (or quotient) of such prime functions with different zeros ζ. An explicit formula, in the form of an infinite product, can be given for $\omega(z,\zeta)$ in terms of the elements of the Schottky group. The product is known to converge if the circles C_i are sufficiently small and well separated. In terms of the prime function, Crowdy was able to give a moderately compact formula for the map from a circle with m disjoint circular holes to a polyhedron with m polyhedral holes.

Alternative Approaches

The methods used by DeLillo, Elcrat, and Pfaltzgraff to derive a formula mapping the exterior of a collection of non-intersecting circles onto the exterior of a similar number of polygons can also be made to yield such a formula. To that end, let $w_{k,i} = f(z_{k,i})$, where $z_{k,i} = c_i + r_i \exp(i\theta_{k,i})$ is the kth prevertex on the ith circle C_i with center c_i and radius r_i, making $w_{k,i}$ the kth corner on the ith target polygon $\Gamma_i = f(C_i)$. With $C_0 = U$, the index i can be allowed to run from 0 to m, while j goes from 0 to ∞, and k goes from 1 to K_i on each circle C_i. The desired mapping is then obtained by quadrature from the fact that:

$$f'(z;\Lambda) = \mathrm{A}\Pi_k\,(z - z_{k,0})^{\beta_{k,0}}$$
$$\Pi_i\,\Pi_j\,\Pi_v\,\{\Pi_k(z - z_{k,v_0})^{\beta_{k,0}}\,\Pi_k(z - z_{k,v_i})^{\beta_{k,i}}\}. \qquad (2)$$

Here v is a multi-index specifying—in a manner that need not concern us here—a sequence of reflections through one after another of the circles $C_0, \ldots, C_m$, which, for each fixed i and j, is to run through all sequences of j reflections not terminating in a reflection through C_i. In practice, that would include a great many sequences indeed if m were not small and if the infinite product in j were not truncated after a small number of terms. Λ is merely a vector containing all the parameters appearing on the right side of (2). DeLillo [3] confirms at length that Crowdy's more compact formula does in fact agree with (2). A possibly unexpected development is the presence of Poincaré θ_2 series in the mapping formulas for domains D of arbitrary connectivity.

The components of Λ include the $m + 1$ centers and $m + 1$ radii of the circles $C_0, \ldots, C_m$, as well as the positions (arguments) of the prevertices $z_{i,k}$ on those circles. Once $c_0 = 0$, $r_0 = 1$, and $\theta_{0,1} = 0$ are chosen, $3m + K_0 + \ldots + K_m - 1$ free real parameters remain to satisfy an equal number of equations specifying the locations of the given vertices $w_{i,k}$.

In yet another paper on the subject, DeLillo et al. launch [4] an attack on the parameter problem patterned on Trefethen's original SCPACK, a Fortran package dating back to the late 1970s. The idea is to choose an initial parameter vector Λ, and to integrate $f'(z;\Lambda)$ along the arcs of the circles C_i joining successive prevertices $z_{i,k} = c_i + r_i\theta_{i,k}$ to obtain initial estimates of the $K_0 + \ldots + K_m$ side lengths $|w_{k+1,i} - w_{k,i}|$, $2m$ centroids, and m rotation angles $\theta_{i,0}$ of the images $\Gamma_i = f_i\,(C_i)$ of the C_i relative to C_0. Then, after judicious adjustment of Λ, the process is repeated to obtain improved estimates, and so on. The authors report that, in several trial cases, convergence to the desired parameters of the desired map has been achieved. This work appears to be in its infancy, with significant improvements in numerical technique still to come. Whether or not Crowdy's prime functions eventually lead to improved numerical mapping methods for multiply connected domains, they have already stimulated activity in a long dormant branch of geometric function theory.

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