Journal of Applied Mathematics and Physics (ZAMP)
Vol. 41, November 1990

0044-2275/90/060755-11 \$ 1.50 + 0.20

Constant vorticity Riabouchinsky flows from a variational principle

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1. Introduction

The computation of two dimensional, inviscid flows past bodies which incorporate vortex sheets and regions of constant vorticity has been the object of extensive recent activity [1], [2], [3]. This problem takes its significance from the proposal of such flows as infinite Reynolds number limits of solutions of the Navier-Stokes equations and as approximations of separated flows [4], [5], [6], [7]. We propose here a method which is based on a variational principle for minimization of an energy functional. The problem dealt with will be the modification of the classical Riabouchinsky flow which was studied recently by Pullin [1] (and earlier by Childress [8] in a slender eddy approximation). For this problem the variational principle which we use is a generalization of the one introduced by Garabedian-Lewy-Schiffer [9], [10] for Riabouchinsky flows. The constant vorticity flow region leads to the addition of a term involving the torsional rigidity of the domain in which this rotational flow occurs. We show in an appendix that this variational principle can be reformulated as one for maximization of the total energy subject to constraints on the impulse, as well as the area of the vorticial region. This establishes a connection between this work and earlier work of Turkington on vortex patches.

The main computation involved in evaluation of the functional which we consider is that of certain Dirichlet integrals over the external and interior regions. This is done by mapping these regions conformally onto the unit disk and using Fourier series (and the invariance of the Dirichlet integral under conformal mapping). We have chosen the Riabouchinsky problem to test our method in order to avoid the computational difficulties associated with cusped boundaries which occur in other recent work [3], [11]. The conformal mapping problems associated with cusped regions create serious computational difficulties; we hope to return to this in a future work. (Near a cusp the domain is "thin", a situation in which the construction of the conformal map from the disk to the domain is difficult due to crowding phenomena.) The choice of this simpler problem allows us to focus attention on the variational principle. This method can be considered as an alternative to the one used by Pullin [1], Moore, Saffman, and Tanveer [3] where a nonlinear integral equation is solved. The use of this variational principle is a slower method since each evaluation of the objective function requires the solution of two Dirichlet problems (using conformal mapping). Our justification for introducing this idea, is to provide an independent solution of an interesting and difficult problem.

2. Formulation of the problem

The problem which we consider is to find an irrotational flow past a pair of symmetrically placed arcs connected by a free streamline. The flow is symmetric about the x-axis, and the interior region, bounded by the x-axis, the obstacles and the free streamline, contains a rotational flow with constant vorticity. There is a constant jump in the squared magnitude of the velocities across the free streamline. This follows from Bernoulli's equation and continuity of the pressure [1]. The flow is also symmetric about the line midway between the obstacles. All this is depicted in Figure 1 below.

If the vorticity is negative, then the flow in Ω_2 along Γ is in the same direction (left to right) as the flow in Ω_1 along Γ; if the vorticity is positive, it is in the opposite direction.

If stream functions are introduced in the exterior (region 1) and the interior (region 2) we obtain the following free boundary problem. (In what follows dimensionless variables have been introduced. We have normalized the velocity at infinity of the exterior flow and the distance between the obstacles and vertical line of symmetry to be one.)

$$\Delta\psi_1 = 0 \quad \text{in } \Omega_1$$

$$\psi_1 = y - \frac{ay}{x^2+y^2} + \cdots \quad \text{at } \infty \tag{1}$$

$$\psi_1 = 0 \quad \text{on } \partial\Omega_1$$

$$-\Delta\psi_2 = \omega \quad \text{in } \Omega_2, \tag{2}$$

$$\psi_2 = 0 \quad \text{on } \partial\Omega_2,$$

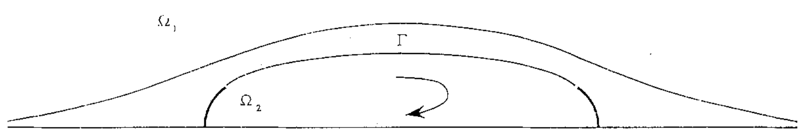

Figure 1
Flow is irrotational in Ω_1, uniform at infinity, rotational with vorticity ω in Ω_2. There is a vortex sheet with strength $1+\sigma$ on Γ.