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Radial and circular slit maps of unbounded multiply connected circle domains

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Infinite product formulae for conformally mapping an unbounded multiply connected circle domain to an unbounded canonical radial or circular slit domain, or to domains with both radial and circular slit boundary components are derived and implemented numerically and graphically. The formulae are generated by analytic continuation with the reflection principle. Convergence of the infinite products is proved for domains with sufficiently well-separated boundary components. Some recent progress in the numerical implementation of infinite product mapping formulae is presented.

Keywords: conformal maps; multiply connected domains; canonical domains

1. Introduction

We develop formulae for conformal maps $f:\Omega \to \mathbb{P}$ from unbounded multiply connected circle domains to canonical unbounded slit domains. A *circle domain* Ω is a domain of connectivity m in the extended complex plane $\mathbb{C}^*$ that contains the point at infinity, and whose m boundary components are circles, C_j, $j=1, \ldots, m$. A *radial* or *circular slit domain* $\mathbb{P}$ is a domain in $\mathbb{C}^*$, $\infty \in \mathbb{P}$ with boundary consisting of m closed segments lying on rays from the origin or m closed circular arcs lying on circles centred at the origin, respectively. Circle, radial slit and circular slit domains are three of the classes of canonical domains in Koebe's classification of multiply connected domains. There are various functional relationships between pairs of slit mappings from different canonical classes (Nehari 1952, Chap. 7), but the circle domains are not related to other canonical classes in such an elementary fashion. Thus, it is of great interest to be able to find explicit formulae for mapping the circle domains onto the radial and the circular slit domains.

We derive our mapping formulae by using the reflection principle to extend the mapping f beyond Ω to a globally defined function. Then, complete knowledge of the zeros and poles of the globally defined function enables one to express f as an

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infinite product. This is a more direct determination of f than the analogous process in finding the Schwarz–Christoffel mapping formula for general polygonal domains where the reflection process leads to a determination of the *derivative* of the mapping function (DeLillo *et al.* 2004). The remaining problem of trying to determine f from an integral of many non-elementary infinite products with unknown accessory parameters is still not solved in a satisfactory general manner (DeLillo *et al.* 2006). Thus, it is quite interesting to see in the present work that for radial and circular slit mappings, the problem of integrating the derivative of the desired mapping function is eliminated. Schwarz–Christoffel formulae for bounded polygonal domains were derived by Crowdy (2005) and for unbounded polygonal domains by Crowdy (2007) using Schottky–Klein prime functions (see also DeLillo 2006).

The techniques in this paper differ from those employed by Crowdy & Marshall (2006). They follow the approach of Schiffer (1950) giving the radial and circular slit maps in terms of Green's functions by using Schottky–Klein prime functions of the circular domains. By contrast, we use directly the properties that such mappings must have and basic reflection arguments to derive our formulae without recourse to Green's functions. There is a close connection between reflections in circles and the Schottky group, which DeLillo (2006) uses to derive relations between the Schottky–Klein prime functions and (bounded and circular) slit maps in the context of Schwarz–Christoffel mapping.

Green's functions for multiply connected domains are useful in many applications. Crowdy & Marshall (2007) have given Green's functions for circle domains in terms of Schottky–Klein prime functions. Our methods can also be used to give explicit formulae for Green's functions for circle domains. As given by Nehari (1952), the radial and circular slit maps are central components in the construction of Green's functions of a given domain. In a similar fashion, the combined circular/radial slit map given below can be used for the construction of the Robin function, the Green's function for the mixed boundary-value problem. With our approach, the maps to (bounded) circular and radial slit discs and annuli are also needed. However, these maps are closely related to the maps given below. The circular slit disc map is already given by DeLillo (2006). Details will appear in a forthcoming article.

The paper is organized as follows. In §2, we give some preliminaries on reflection in circles. In §3, we give simple derivations of infinite product formulae for maps $w=f(z)$ from circle domains to canonical radial, circular and combined radial/circular slit domains. We prove convergence of the formulae for domains satisfying a separation condition. The evaluation of the (truncated) product formula based on successive reflections is very inefficient. Therefore, in §4, we give an efficient method for calculating the slit maps based on solving a least-squares problem, as given by Trefethen (2005; see also Finn *et al.* 2003).

2. Preliminaries

We shall introduce notation, recall basic facts about reflections in circles and relate useful information. As already mentioned, $w=f(z)$ is a conformal map $f:\Omega\to\mathbb{P}$ from a circle domain to a slit domain of connectivity m, with c_j and r_j denoting the centres and radii, respectively, of the mutually exterior boundary circles C_j.

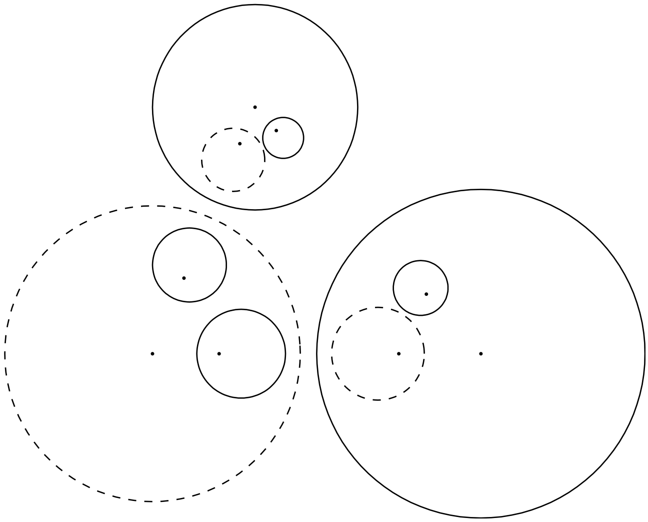

Figure 1. $N=1$ levels of reflected circles and centres for unbounded case.

The *reflection* of z through a circle C with centre c and radius r is given by

$$z^* = \rho_C(z) := c + \frac{r^2}{\bar{z} - \bar{c}}, \tag{2.1}$$

i.e. z and z^* are *symmetric points with respect to the circle* C. If $C = C_\tau$, where τ is an index of a circle, we will denote ρ_{C_τ} by ρ_τ. In our work, the function f that maps Ω on to a canonical slit domain is known to exist by classical uniformization results of Koebe, but no formula is provided by the non-constructive proofs. We develop our mapping formulae by using analytic continuation with the reflection principle to extend f onto $\mathbb{C}^* \backslash \{\text{lim. pts.}\}$. Briefly, we begin by continuing f beyond Ω with reflection across its m boundary circles, C_j. We repeat this process, reflecting across the $m(m-1)$ reflections of the original m boundary circles thereby producing $m(m-1)^2$ additional reflected circles. Unlimited iteration of this process produces a global extension, $\hat{f}$, of f. The values of the extension are obtained at each reflection by reflecting the values of the already defined function across the appropriate boundary slit. Then it can be seen that the global $\hat{f}$ is characterized by its zeros and poles and that the formula for f in terms of infinite products of these zeros and poles follows. It is useful to note that the number of new regions and new boundary components created by the reflections at a given level is $m-1$ times that at the preceding level.

We need the notation of multi-indices to denote reflected domains and boundary circles. When Ω is reflected through the boundary circle C_1, it produces a domain $\Omega_1 := \rho_1(\Omega)$ inside C_1 that is bounded by C_1 and the reflections of $C_2, \ldots, C_m$, which we denote $C_{12}, C_{13}, \ldots, C_{1m}$, respectively, i.e. $C_{1j} = \rho_1(C_j)$. Similarly, $\Omega_k = \rho_k(\Omega)$ and $C_{kj} = \rho_k(C_j)$, $j \neq k$. Figure 1 illustrates reflections of circles and centres for $m=3$ for one level ($N=1$, below). In general, for a multi-index, $\nu = \nu_1 \nu_2 \cdots \nu_n$ and a quantity Q (point, circle or region),

$$Q_\nu = \rho_\nu(Q) = \rho_{\nu_1} \rho_{\nu_2} \cdots \rho_{\nu_n}(Q) = \rho_{\nu_1}(\rho_{\nu_2}(\cdots(\rho_{\nu_n}(Q))\cdots)). \tag{2.2}$$

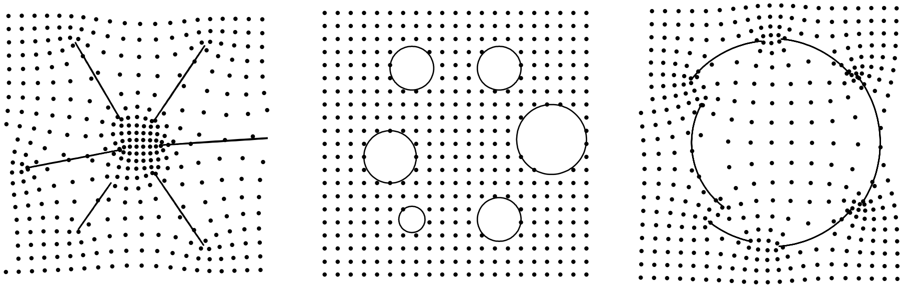

Figure 2. Maps from unbounded circle domains to radial and circular slit domains with a Cartesian grid. The products are truncated to $N=4$ levels of reflection. With $N=3$ the slits appear to be slightly open as in figure 3.

Definition 2.1. The set of multi-indices of length n will be denoted

$$\sigma_n = \{\nu_1\nu_2\ldots\nu_n : 1 \le \nu_j \le m, \nu_k \ne \nu_{k+1}, k = 1, \ldots, n-1\}, \quad n > 0, \tag{2.3}$$

and $\sigma_0=\phi$ (in which case $\nu i=i$ for $\nu\in\sigma_0$ below). Also

$$\sigma_n(i) = \{\nu \in \sigma_n : \nu_n \ne i\}, \tag{2.4}$$

denotes the set of sequences in σ_n whose last factor never equals i.

From propostion 1 of DeLillo *et al.* (2004), we also have that if $\nu\in\sigma_n$, $n>1$, then $\Omega_\nu = \rho_\nu\left(\Omega_{\nu_1\ldots\nu_{n-1}}\right)$ is a circular domain with outer boundary C_ν and $m-1$ interior boundary circles. Clearly, σ_n contains $m(m-1)^{n-1}$ elements, which is consistent with our earlier comment that the number of circular domains Ω_ν at a particular level of reflections, say $\nu\in\sigma_n$, is $m-1$ times the number of domains $\Omega_{\tilde\nu}$, $\tilde\nu \in \sigma_{n-1}$, at the preceding level.

In order to state our convergence results, we need the following definition and lemma. The *separation parameter* of the region is

$$\Delta := \max_{i,j;i\ne j}\frac{r_i + r_j}{|c_i - c_j|} < 1, \quad 1 \le i, \quad j \le m, \tag{2.5}$$

for the assembly of m mutually exterior circles that form the boundary of Ω (cf. Henrici 1986, p. 501). Let $\tilde C_j$ denote the circle with centre c_j and radius r_j/Δ. Then geometrically, $1/\Delta$ is the smallest magnification of the m radii such that at least two $\tilde C_j$'s just touch. We will use the following inequality from Henrici (1986; p. 505):

Lemma 2.2.

$$\sum_{\nu\in\sigma_{n+1}} r_\nu^2 \le \Delta^{4n}\sum_{i=0}^{m} r_i^2. \tag{2.6}$$

3. Maps to the canonical radial and circular slit domains

In this section, we use simple reflection arguments to derive the mappings of unbounded circle domains to the canonical radial and circular slit domains as well as mapping to a domain with both radial and circular slit boundary components (figures 2 and 3).

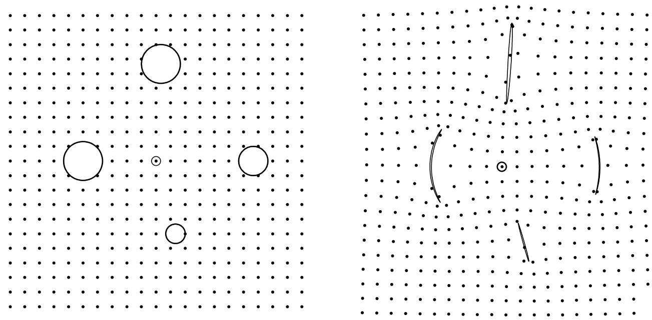

Figure 3. Map to circular/radial slit domain with $N=3$ levels of reflection.

(a) Radial slit maps

We will give a detailed derivation and proof of the formula for a conformal mapping f of an m-connected circle domain Ω onto a radial slit domain $\mathbb{P}$ with $f(a)=0$ and $f(\infty)=\infty$. We begin with a brief outline of the procedure. First, we extend f to a globally defined (many valued) $\hat{f}$ on $\mathbb{C}^*\backslash\{\text{lim. pts.}\}$ by repeated use of the reflection principle. When f or $\hat{f}$ is reflected across a circle C, the corresponding extension of $w=f(z)$ across the radial slit $\gamma=f(C)$ at angle θ is given by reflecting w across γ to $w^*=\mathrm{e}^{\mathrm{i}2\theta}\bar{w}$. The latter reflection leaves the w-values zero and infinity fixed and hence the zero set of the extended f will be the point a and all of its reflections, and similarly for ∞ and the other the poles of $\hat{f}$ Thus it seems plausible to think that the mapping $f:\Omega\rightarrow\mathbb{P}$ can be expressed by a formula

$$f(z) = (z-a)\prod_{k=1}^{m}\prod_{\substack{j=0\\ \nu\in\sigma_j(k)}}^{\infty}\frac{z-\rho_\nu(a_k)}{z-\rho_\nu(c_k)}, \tag{3.1}$$

where the a_k's are the reflections of a across the boundary circles C_k and c_k's are the reflections of ∞ across the boundary circles of Ω. Further details including convergence will be proven when the m circles with centres c_k satisfy our separation condition in theorem 3.2. Note that $f(a)=0$ and $f(z)/z=O(1)$ near ∞.

It is important to note that, although the global $\hat{f}$ is many valued, the differential expression $\hat{f}'(z)/\hat{f}(z)$ is single valued. Indeed, any two values, $\hat{f}_r(z)$ and $\hat{f}_s(z)$ of $\hat{f}$ at a point $z\in\mathbb{C}^*\backslash\{\text{lim. pts.}\}$ are related by the composition of an even number of reflections in lines and hence $\hat{f}_s(z)=A\hat{f}_r(z)$ for some $A\in\mathbb{C}$. The differential expression, $f'(z)/f(z)$, is invariant under maps $w\mapsto Aw$, i.e. $(af(z))'/(af(z))=f'(z)/f(z)$. Thus, if one begins with $f'(z)/f(z)$ in Ω, the reflection process yielding the many-valued $\hat{f}$ also defines a global analytic function, $\hat{f}'(z)/\hat{f}(z)$, that is defined and *single valued* on $\mathbb{C}^*\backslash\{\text{lim. pts.}\}$. We shall refer to $S(z)=f'(z)/f(z)$ as the *singularity function*. Our proof will depend on showing that

$$S(z) = \frac{1}{z-a}+\sum_{k=1}^{m}\sum_{\substack{j=0\\ \nu\in\sigma_j(k)}}\left(\frac{1}{z-\rho_\nu(a_k)}-\frac{1}{z-\rho_\nu(c_k)}\right), \tag{3.2}$$

or in convergent form,

$$S(z) = \frac{1}{z-a} + \sum_{k=1}^{m} \sum_{\substack{j=0 \\ \nu\in\sigma_j(k)}}^{\infty} \left(\frac{\rho_\nu(a_k) - \rho_\nu(c_k)}{(z-\rho_\nu(a_k))(z-\rho_\nu(c_k))} \right). \tag{3.3}$$

Our task is to show that, indeed, $f'(z)/f(z) = S(z)$. Note that

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + \left[\frac{1}{z^2}\right] \quad \text{and} \quad S(z) = \frac{1}{z} + \left[\frac{1}{z^2}\right]. \tag{3.4}$$

We will show that the sums truncated to N levels of reflection,

$$S_N(z) = \frac{1}{z-a} + \sum_{k=1}^{m} \sum_{\substack{j=0 \\ \nu\in\sigma_j(k)}}^{N} \left(\frac{1}{z-\rho_\nu(a_k)} - \frac{1}{z-\rho_\nu(c_k)} \right), \tag{3.5}$$

converge uniformly to $S(z)$ for $z \in \overline{\Omega}$ as $N \to \infty$, provided the circles satisfy our separation condition, that $S(z)$ satisfies an appropriate boundary condition, and that $f(z) = \exp(\int S(z)\,\mathrm{d}z)$, our main theorem.

Our boundary conditions are given by

Lemma 3.1. $\mathrm{Re}\{(z-c_k)f'(z)/f(z)\} = 0$, $z \in C_k$.

Proof. For $z \in C_k$, we have $z = c_k + r_k \mathrm{e}^{\mathrm{i}\theta}$ and since $f(z)$ maps to radial slits, we have $\arg f(z) = \text{const}$. Therefore,

$$0 = \frac{\partial}{\partial\theta} \arg f(z) = \frac{\partial}{\partial\theta} \mathrm{Im} \log f(c_k + r_k \mathrm{e}^{\mathrm{i}\theta}) = \mathrm{Im}\, \mathrm{i} r_k \mathrm{e}^{\mathrm{i}\theta} \frac{f'}{f} = \mathrm{Re}\, r_k \mathrm{e}^{\mathrm{i}\theta} \frac{f'}{f}(c_k + r_k \mathrm{e}^{\mathrm{i}\theta}). \tag{3.6}$$

■

We now state our main theorem for radial slit maps.

Theorem 3.2. *Let* $\mathbb{P}$ *be an unbounded m-connected radial slit region,* $0, \infty \in \mathbb{P}$, *and* Ω *a conformally equivalent circular domain,* $a, \infty \in \Omega$. *Furthermore, suppose* Ω *satisfies the separation property* $\Delta < (m-1)^{-1/4}$ *for* $m > 1$. *Then* Ω *is mapped conformally onto* $\mathbb{P}$ *by f with* $f(a) = 0$ *and* $f(\infty) = \infty$ *if and only if*

$$f(z) = C(z-a) \prod_{k=1}^{m} \prod_{\substack{j=0 \\ \nu\in\sigma_j(k)}}^{\infty} \frac{z-\rho_\nu(a_k)}{z-\rho_\nu(c_k)}, \tag{3.7}$$

for some constant C. ■

Proof. The proof, that a map f to a radial slit domain must necessarily be of the form (3.7), follows very closely the proof of theorem 1 by DeLillo *et al.* (2004). The central idea is to prove that $f'(z)/f(z) = S(z)$ by means of the argument principle. We shall use the following two results whose proofs are given after the present proof in order to keep the essence of the present proof from being obscured by calculation details.

(i) Convergence: $S(z) = \lim_{N\to\infty} S_N(z)$ uniformly on $\overline{\Omega}$.

(ii) Boundary conditions: $\mathrm{Re}\,\{(z-s_j)S(z)\}_{z\in C_j} = 0, j = 1, \ldots, m$.

For $z \in \overline{\Omega}$, we define the functions

$$H(z) := \int^z S(\zeta)\,\mathrm{d}\zeta, \quad H_N(z) := \int^z S_N(\zeta)\,\mathrm{d}\zeta, \quad P(z) := \mathrm{e}^{H(z)}. \tag{3.8}$$

We first note that

$$H_N(z) = \int^z S_N(\zeta)\,\mathrm{d}\zeta = \sum_{j=0}^{N}\sum_{i=1}^{m}\sum_{\nu\in\sigma_j(i)} \int^z \frac{1}{\zeta - a_{\nu i}} - \frac{1}{\zeta - s_{\nu i}}\,\mathrm{d}\zeta, \tag{3.9}$$

is defined and analytic in Ω since its periods are zero. Indeed $\int_{C_\tau+} S_N(z)\,\mathrm{d}z = 0$, $\tau = 1, \ldots, m$, where $C_\tau+$ is a circle concentric with the boundary circle C_τ with radius slightly larger than that of C_τ since the residues add out in pairs. Furthermore, $H(z)$ is analytic in Ω since

$$H(z) = \lim_{N\to\infty} H_N(z) = \lim_{N\to\infty} \int^z S_N(\zeta)\,\mathrm{d}\zeta = \int^z S(\zeta)\,\mathrm{d}\zeta, \quad z \in \overline{\Omega}, \tag{3.10}$$

with $S_N(z) \to S(z)$ uniformly on closed subsets of $\overline{\Omega}$.

The next step is to develop a formula for the antiderivative (up to an additive constant)

$$\begin{aligned} H_N(z) &= \int^z S_N(\zeta)\,\mathrm{d}\zeta = \sum_{j=0}^{N}\int^z \sum_{i=1}^{m}\sum_{\nu\in\sigma_j(i)} \left(\frac{1}{\zeta - a_{\nu i}} - \frac{1}{\zeta - s_{\nu i}}\right)\mathrm{d}\zeta \\ &= \sum_{j=0}^{N}\sum_{i=1}^{m}\sum_{\nu\in\sigma_j(i)} \int^z \frac{1}{\zeta - a_{\nu i}} - \frac{1}{\zeta - s_{\nu i}}\,\mathrm{d}\zeta \\ &= \sum_{j=0}^{N}\sum_{i=1}^{m}\sum_{\nu\in\sigma_j(i)} \log\left(\frac{z - a_{\nu i}}{z - s_{\nu i}}\right) \\ &= \sum_{i=1}^{m} \sum_{\substack{j=0 \\ \nu\in\sigma_j(i)}}^{N} \log\left(\frac{z - a_{\nu i}}{z - s_{\nu i}}\right), \end{aligned} \tag{3.11}$$

where each logarithm is the branch that vanishes at $z = \infty$, i.e. $\log 1 = 0$. From the preceding formula, one has

$$P(z) = \lim_{N\to\infty} \exp\{H_N(z)\} = \lim_{N\to\infty} \prod_{\substack{j=0 \\ \nu\in\sigma_j(i)}}^{N} \left(\frac{z - a_{\nu i}}{z - s_{\nu i}}\right), \tag{3.12}$$

and hence the product formula for $P(z)$,

$$P(z) = \mathrm{e}^{H(z)} = \prod_{i=1}^{m} \prod_{\substack{j=0 \\ \nu\in\sigma_j(i)}}^{\infty} \left(\frac{z - a_{\nu i}}{z - s_{\nu i}}\right). \tag{3.13}$$

Our theorem, $f(z) = A\int^z P(\zeta)\,\mathrm{d}\zeta + B$, is equivalent to showing that the quotient

$$Q(z) := \frac{f(z)}{P(z)} \equiv \text{const.} \tag{3.14}$$

To accomplish this, we will apply the argument principle to $Q(z)$. First, observe that $P'(z)=H'(z)\mathrm{e}^{H(z)}=S(z)P(z)$, i.e. $P'(z)/P(z)=S(z)$, and

$$Q'=\frac{f}{P}\left(\frac{f'}{f}-\frac{P'}{P}\right)=Q\left(\frac{f'}{f}-S\right). \tag{3.15}$$

Then, for $z=c_j+r_j\mathrm{e}^{\mathrm{i}\theta}\in C_j$, the boundary conditions of lemma 3.1 and theorem 3.4 on f'/f and S, respectively, give

$$\begin{aligned}\frac{\partial}{\partial\theta}\arg Q(z)&=\frac{\partial}{\partial\theta}\operatorname{Im}\{\log Q(z)\}=\operatorname{Re}\left\{(z-c_j)\frac{Q'(z)}{Q(z)}\right\}\\&=\operatorname{Re}\left\{(z-c_j)\left(\frac{f'(z)}{f(z)}-S(z)\right)\right\}=0.\end{aligned} \tag{3.16}$$

By our construction of $S(z)$, $f'(z)/f(z)-S(z)$ is continuous on all C_j. Therefore, arg Q is constant on each of the m boundary circles, C_j. Equivalently, $Q(C_j)$, the image of C_j, lies on a half-ray emanating from the origin. It is clear by the local behaviour of f and formula (3.13) that $Q=f/P$ is continuous on each C_j and not equal to 0 or ∞ there, since $f,P\neq 0$. Thus, for any $w_0\in\mathbb{C}\backslash Q(C_j)$, $j=1,\ldots,m$, the winding number of $Q(C_j)$ around w_0, $n(Q(C_j),w_0)=0$ for all j. Let C_R be a large circle of radius R centred at the origin and containing w_0 and all the C_j's in its interior, and write $C=C_1\cup\cdots\cup C_m\cup C_R$ with the curves oriented so that the region interior to C_R and exterior to the C_j's is on the left. Since Q has no poles in the region, by the argument principle (for bounded regions), the number of times $Q(z)$ assumes the value w_0 is

$$n(Q(C),w_0)=n(Q(C_1),w_0)+\cdots+n(Q(C_m),w_0)+n(Q(C_R),w_0)=n(Q(C_R),w_0). \tag{3.17}$$

We now will show that $n(Q(C_R),w_0)=0$. First,

$$n(Q(C_R),w_0)=\frac{1}{2\pi i}\int_{|z|=R}\frac{Q'(z)}{Q(z)-w_0}\,\mathrm{d}z=\frac{1}{2\pi i}\int_{|z|=R}\frac{Q'(z)/Q(z)}{1-w_0/Q(z)}\,\mathrm{d}z. \tag{3.18}$$

Recall that $Q'(z)/Q(z)=f'(z)/f(z)-S(z)=(1/z)+[1/z^2]-(1/z)+[1/z^2]=O(1/z^2)$ for z near ∞, and that, $Q(\infty)=f(\infty)/P(\infty)$ is a finite constant. It suffices to assume $w_0\neq Q(\infty)$. Then $w_0\neq Q(z)$ for R sufficiently large and there are constants $A,B>0$ such that

$$\left|\int_{|z|=R}\frac{Q'(z)/Q(z)}{1-w_0/Q(z)}\,\mathrm{d}z\right|\leq A\int_{|z|=R}|Q'(z)/Q(z)||\,\mathrm{d}z|\leq B\int_0^{2\pi}\frac{1}{R^2}R\,\mathrm{d}\theta\to 0, \tag{3.19}$$

as $R\to\infty$. Therefore, $n(Q(C),w_0)=0$ and $Q(z)\neq w_0$ for $w_0\notin Q(C_j)$ and $w_0\neq Q(\infty)$. Thus, Q assumes values only on the radial segments $Q(C_j)$ (or $Q(\infty)$) and hence, by the open mapping property of analytic functions, Q must be constant on Ω.

Finally, we show that a function $w=f(z)$ of the form (3.7) always determines a conformal map to the conformally equivalent slit domain $\mathbb{P}$ with $f(a)=0$ when Ω satisfies the separation property: by the basic existence theorem for maps of multiply connected domains, Ω is conformally equivalent, via a map g with $g(a)=0$, to some radial slit domain $\mathbb{P}'$. By the above argument, $g(z)$ must have the form (3.7), and by uniqueness of the conformal maps we must have $\mathbb{P}'=C\mathbb{P}$ for some constant C. ∎

In the special case when $m=2$, there is no restrictive separation hypothesis; since then $\Delta<(m-1)^{-1/4}=1$ is equivalent to the fact that the two boundary components are disjoint.

(i) Convergence of $S(z)$

For $j = 0, 1, 2, \ldots$, we write

$$A_j(z) = \sum_{i=1}^{m} \sum_{\nu \in \sigma_j(i)} \left(\frac{1}{z - a_{\nu i}} - \frac{1}{z - s_{\nu i}} \right) = \sum_{i=1}^{m} \sum_{\nu \in \sigma_j(i)} \frac{a_{\nu i} - s_{\nu i}}{(z - a_{\nu i})(z - s_{\nu i})}, \tag{3.20}$$

and hence, in brief notation,

$$S_N(z) = \sum_{j=0}^{N} A_j(z), \quad S(z) = \lim_{N \to \infty} S_N(z). \tag{3.21}$$

Let

$$\delta = \delta_\Omega = \inf_{z \in \Omega} \{|z - a_\nu|, |z - s_\nu| : k = 1, \ldots, m, \nu \in \sigma\}. \tag{3.22}$$

Then, clearly $\delta > 0$ holds since the a_ν's and the S_ν's lie inside the circles.
We have the following

Theorem 3.3. *For connectivity* $m \geq 2$, $S_N(z)$ *converges to* $S(z)$ *uniformly on* $\overline{\Omega}$ *satisfying the following estimate*

$$|S(z) - S_N(z)| = O((\mu^2 \sqrt{m-1})^{N+1}), \tag{3.23}$$

for regions satisfying the separation condition

$$\Delta < \frac{1}{(m-1)^{1/4}}. \tag{3.24}$$

Proof. Note that the number of terms in the $A_j(z)$ sum is $O((m-1)^j)$. This exponential increase in the number of terms is the principal difficulty in establishing convergence. Recall that $r_{\nu i}$ is the radius of circle $C_{\nu i}$. We bound $A_j(z)$ for $z \in \overline{\Omega}$ by using the facts $|a_{\nu i} - s_{\nu i}| < 2r_{\nu i}$, and the Cauchy–Schwarz inequality, as follows:

$$\begin{aligned}
|A_j(z)| &\leq \sum_{\nu \in \sigma_j(i)} \sum_{i=1}^{m} \frac{|a_{\nu i} - s_{\nu i}|}{|z - a_{\nu i}||z - s_{\nu i}|} \\
&\leq \frac{2}{\delta^2} \sum_{\nu \in \sigma_j(i)} \sum_{i=1}^{m} r_{\nu i} \\
&\leq \frac{2}{\delta^2} \left(\sum_{\nu \in \sigma_j(i)} \sum_{i=1}^{m} r_{\nu i}^2 \right)^{1/2} \left(\sum_{\nu \in \sigma_j(i)} \sum_{i=1}^{m} 1 \right)^{1/2} \\
&= \frac{2}{\delta^2} \left(\sum_{\nu \in \sigma_j(i)} \sum_{i=1}^{m} r_{\nu i}^2 \right)^{1/2} \sqrt{m}(m-1)^{j/2} \\
&\leq \frac{2}{\delta^2} \Delta^{2j} \left(\sum_{i=1}^{m} r_i^2 \right)^{1/2} \sqrt{m}(m-1)^{j/2} \\
&\leq C \Delta^{2j} (m-1)^{j/2},
\end{aligned} \tag{3.25}$$

by lemma 2.2 where $\delta = \delta_\Omega$. Therefore, the series converges if $\Delta^2 \sqrt{m-1} < 1$. ■

(ii) $S(z)$ satisfies the boundary condition

Here, we prove that $S(z)$ satisfies the boundary condition

$$\mathrm{Re}\,\{(z-s_j)S(z)\}_{z\in C_j}=0, \tag{3.26}$$

as claimed in the proof of the main theorem. We will use the formula

$$\mathrm{Re}\left\{\frac{w}{w-1}+\frac{w^*}{w^*-1}\right\}=1, \tag{3.27}$$

where w and $w^*=1/\bar{w}$ are symmetric points with respect to the unit circle.

The following theorem shows, for general m, that $S(z)$ satisfies the boundary condition for $f'(z)/f(z)$.

Theorem 3.4. *If $\Delta<(m-1)^{-1/4}$, then for $z\in C_i$*

$$\mathrm{Re}\,\{(z-s_i)S_N(z)\}=O((\Delta^2\sqrt{m-1})^N), \tag{3.28}$$

and

$$\mathrm{Re}\,\{(z-s_i)S(z)\}=0. \tag{3.29}$$

Proof. The idea of the proof is, for $z\in C_p$, to use properties of the reflections (2.2) to group terms in $S_N(z)$ related by reflection ρ_p through C_p with $z\in C_p$ as follows:

$$\begin{aligned}S_N(z)=&\left[-\frac{1}{z-c_p}+\left(\frac{1}{z-a}+\frac{1}{z-a_p}\right)\right]+\cdots\\&+\left[\left(\frac{1}{z-a_\nu}+\frac{1}{z-a_{p\nu}}\right)-\left(\frac{1}{z-s_\nu}+\frac{1}{z-s_{p\nu}}\right)\right]+\cdots.\end{aligned} \tag{3.30}$$

Then, multiplying by $z-c_p$, we have in more detail,

$$\begin{aligned}(z-s_p)S_N(z)=&-1+\frac{(z-s_p)/(a-s_p)}{(z-s_p)/(a-s_p)-1}+\frac{(z-s_p)/(a_p-s_p)}{(z-s_p)/(a_p-s_p)-1}\\&+\sum_{j=0}^{N-1}\sum_{i=1}^{m}\sum_{\substack{\nu\in\sigma_j(i),\\ \nu i,\nu_1\neq p}}\left(\frac{(z-s_p)/(a_{\nu i}-s_p)}{(z-s_p)/(a_{\nu i}-s_p)-1}+\frac{(z-s_p)/(\rho_p(a_{\nu i})-s_p)}{(z-s_p)/(\rho_p(a_{\nu i})-s_p)-1}\right)\\&-\sum_{j=0}^{N-1}\sum_{i=1}^{m}\sum_{\substack{\nu\in\sigma_j(i),\\ \nu i,\nu_1\neq p}}\left(\frac{(z-s_p)/(s_{\nu i}-s_p)}{(z-s_p)/(s_{\nu i}-s_p)-1}+\frac{(z-s_p)/(\rho_p(s_{\nu i})-s_p)}{(z-s_p)/(\rho_p(s_{\nu i})-s_p)-1}\right)\\&+(z-s_p)\sum_{\substack{j=1,\\ j\neq p}}^{m}\sum_{i=1}^{m}\sum_{j\nu\in\sigma_N(i)}\left(\frac{a_{j\nu i}-s_{j\nu i}}{(z-a_{j\nu i})(z-s_{j\nu i})}\right).\end{aligned} \tag{3.31}$$

We take the real part of the above expression and using, for instance, $w=(z-s_p)/(a_\nu-s_p)$ and noting that $w^*=(z-s_p)/(\rho_p(a_\nu)-s_p)$ (3.27) gives

$$\mathrm{Re}\left\{\frac{(z-s_p)/(a_\nu-s_p)}{(z-s_p)/(a_\nu-s_p)-1}+\frac{(z-s_p)/(\rho_p(z_\nu)-s_p)}{(z-s_p)/(\rho_p(a_\nu)-s_p)-1}\right\}$$

$$=\mathrm{Re}\left\{\frac{w}{w-1}+\frac{w^*}{w^*-1}\right\}=1. \tag{3.32}$$

Taking the real part of (3.31), we see that the first three lines sum to 0. The final $m-1$ terms, all lying inside circles C_i, $i\neq p$, approximate the truncation error and are estimated by

$$\sum_{\nu\in\sigma_{n+1}} r_\nu^2 \le \Delta^{4N}\sum_{i=1}^{m} r_i^2. \tag{3.33}$$

This gives our final result

$$\mathrm{Re}\{(z-s_p)S_N(z)\} = O(\sqrt{m-1}(\Delta^{2N}(m-1)^{N/2}). \tag{3.34}$$

■

(*b*) *Circular slit maps*

The derivation of the map, $w=f(z)$ from an unbounded circle domain to the conformally equivalent unbounded circular slit domain is similar to that of the radial slit domain. This map is closely related to the Green's function for the Dirichlet boundary-value problem. Once again $f(a)=0$ and $f(\infty)=\infty$ with $f(z)\sim z, z\approx\infty$. Again, a_i is the reflection of a across circle C_i and $c_i=s_i$, the centre of circle C_i, is the reflection of ∞ across C_i. In the w-plane, 0 and ∞ just reflect back and forth to each other. Therefore, when we extend f, we will have $f(a_i)=\infty$ and $f(s_i)=0$. In this way, we see that all odd numbers of reflections $a_{\nu_o i}:=\rho_{\nu_o}(a_i)$, $|\nu_o|=2k+1$ of a_i and all even numbers of reflections $s_{\nu_e i}:=\rho_{\nu_e}(c_i)$, $|\nu_o|=2k$ of c_i will be simple zeros, $f(a_{\nu_o i})=f(s_{\nu_e i})=0$. Likewise, all odd numbers of reflections $s_{\nu_o i}, |\nu_o|=2k+1$ of c_i and all even numbers of reflections $a_{\nu_e i}, |\nu_o|=2k$ of a_i will be simple poles, $f(a_{\nu_e i})=f(s_{\nu_o i})=\infty$. The infinite product for $w=f(z)$ therefore has the form,

$$f(z) = (z-a)\prod_{i=1}^{m}\prod_{\substack{j=0\\ \nu_e,\nu_o\in\sigma_j(i)}}^{\infty}\frac{(z-\rho_{\nu_o}(a_i))(z-\rho_{\nu_e}(c_i))}{(z-\rho_{\nu_e}(a_i))(z-\rho_{\nu_o}(c_i))}. \tag{3.35}$$

(where reflections back to a or ∞ are excluded from the product) with $f(a)=0$, provided the m circles with centres c_k satisfy our standard separation criterion.

Now note that, if a circular slit in the w-plane is at radius r_1, then w reflects to $r_1^2/\overline{w}$. Reflection through another circular slit with radius r_2 will then take w to $(r_2/r_1)^2 w$, and so on. Therefore, an even number successive reflection through circular slits will take $w=f(z)$ to $Aw=Af(z)$, for some A real. As a result, the extended function $f'(z)/f(z)=Af'(z)/Af(z)$ is invariant under even numbers of reflections and hence is single valued. Here, our singularity function, in non-convergent form, will be

$$S(z) = f'(z)/f(z) = \frac{\mathrm{d}}{\mathrm{d}z}\log f(z)$$
$$= \frac{1}{z-a} + \sum_{i=1}^{m} \sum_{\substack{j=0 \\ \nu_e,\nu_o \in \sigma_j(i)}}^{\infty} \left(\frac{1}{z-a_{\nu_o i}} - \frac{1}{z-s_{\nu_o i}}\right) + \left(\frac{1}{z-s_{\nu_e i}} - \frac{1}{z-a_{\nu_e i}}\right), \quad (3.36)$$

or in convergent form,

$$S(z) = \frac{1}{z-a} + \sum_{i=1}^{m} \sum_{\substack{j=0 \\ \nu_e,\nu_o \in \sigma_j(i)}}^{\infty} \left(\frac{a_{\nu_o i} - s_{\nu_o i}}{(z-a_{\nu_o i})(z-s_{\nu_o i})}\right) + \left(\frac{s_{\nu_e i} - a_{\nu_e i}}{(z-s_{\nu_e i})(z-a_{\nu_e i})}\right). \quad (3.37)$$

Again, our task is to show that $f'(z)/f(z) = S(z)$. Note that, again,

$$\frac{f'(z)}{f(z)} = \frac{1}{z} + \left[\frac{1}{z^2}\right] \quad \text{and} \quad S(z) = \frac{1}{z} + \left[\frac{1}{z^2}\right]. \quad (3.38)$$

We will show that the sums truncated to N levels of reflection,

$$S_N(z) = \frac{1}{z-a} + \sum_{i=1}^{m} \sum_{\substack{j=0 \\ \nu_e,\nu_o \in \sigma_j(i)}}^{N} \left(\frac{1}{z-a_{\nu_o i}} - \frac{1}{z-s_{\nu_o i}}\right) + \left(\frac{1}{z-s_{\nu_e i}} - \frac{1}{z-a_{\nu_e i}}\right), \quad (3.39)$$

converge uniformly to $S(z)$ for $z \in \overline{\Omega}$ as $N \to \infty$, provided the circles satisfy our separation condition, that $S(z)$ satisfies an appropriate boundary condition, and that $f(z) = \exp(\int S(z)\,\mathrm{d}z)$, our main theorem.

Our boundary conditions are given by

Lemma 3.5. $\mathrm{Im}\{(z-c_k)f'(z)/f(z)\} = 0, z \in C_k$.

Proof. For $z \in C_k$, we have $z = c_k + r_k e^{i\theta}$ and since $f(z)$ maps to circular slits, we have $\log|f(z)| = \mathrm{Re}\log f(z)$=const. Therefore,

$$0 = \frac{\partial}{\partial\theta}\mathrm{Re}\log f(z) = \frac{\partial}{\partial\theta}\mathrm{Re}\log f(c_k + r_k e^{i\theta}) = \mathrm{Re}\, i r_k e^{i\theta}\frac{f'}{f} = -\mathrm{Im}\, r_k e^{i\theta}\frac{f'}{f}(c_k + r_k e^{i\theta}). \quad (3.40)$$

■

We now state our main theorem for circular slit maps.

Theorem 3.6. *Let $\mathbb{P}$ be an unbounded m-connected circular slit region, $0,\infty \in \mathbb{P}$, and Ω a conformally equivalent circular domain, $a,\infty \in \Omega$. Furthermore, suppose Ω satisfies the separation property $\Delta < (m-1)^{-1/4}$ for $m>1$. Then Ω is mapped conformally onto $\mathbb{P}$ by f with $f(a)=0$ and $f(\infty)=\infty$ if and only if*

$$f(z) = (z-a)\prod_{i=1}^{m} \prod_{\substack{j=0 \\ \nu_e,\nu_o \in \sigma_j(i)}}^{\infty} \frac{(z-\rho_{\nu_o}(a_i))(z-\rho_{\nu_e}(c_i))}{(z-\rho_{\nu_e}(a_i))(z-\rho_{\nu_o}(c_i))}, \quad (3.41)$$

for some constant C.

Proof. The proof follows the argument for the radial slit case, using the modified convergence theorems and the following boundary conditions for circular slits. ■

(i) Convergence of $S(z)$

For $j=0,1,2,\ldots$, we write

$$A_j(z) = \sum_{i=1}^{m} \sum_{\nu_e,\nu_o\in\sigma_j(i)} \left(\frac{a_{\nu_o i}-s_{\nu_o i}}{(z-a_{\nu_o i})(z-s_{\nu_o i})}\right) + \left(\frac{s_{\nu_e i}-a_{\nu_e i}}{(z-s_{\nu_e i})(z-a_{\nu_e i})}\right), \quad (3.42)$$

and hence, in brief notation,

$$S_N(z) = \sum_{j=0}^{N} A_j(z), \quad S(z) = \lim_{N\to\infty} S_N(z). \quad (3.43)$$

Let

$$\delta = \delta_\Omega = \inf_{z\in\Omega} \{|z-a_\nu|, |z-s_\nu| : k=1,\ldots,m, \nu\in\sigma\}. \quad (3.44)$$

Then, clearly $\delta<0$ holds since the a_ν's and the S_ν's lie inside the circles.

The convergence of $S_N(z)$ to $S(z)$ is identical to theorem 3.3 for the radial case. The details of the proof are nearly identical and we omit them.

(ii) $S(z)$ satisfies the boundary condition.

Here, we prove that $S(z)$ satisfies the boundary condition

$$\mathrm{Im}\{(z-s_j)S(z)\}_{z\in C_j} = 0, \quad (3.45)$$

as claimed in the proof of the main theorem. We will use the formula

$$\mathrm{Im}\left\{\frac{w}{w-1} - \frac{w^*}{w^*-1}\right\} = 0, \quad (3.46)$$

where w and $w^*=1/\bar{w}$ are symmetric points with respect to the unit circle.

The following theorem shows, for general m, that $S(z)$ satisfies the boundary condition for $f'(z)/f(z)$.

Theorem 3.7. *If* $\Delta<(m-1)^{-1/4}$, *then for* $z\in C_i$

$$\mathrm{Im}\,\{(z-s_i)S_N(z)\} = O((\Delta^2\sqrt{m-1})^N), \quad (3.47)$$

and

$$\mathrm{Im}\,\{(z-s_i)S(z)\} = 0. \quad (3.48)$$

Proof. The idea of the proof is, for $z\in C_p$, to again use the properties of reflections (2.2) to group terms in $S_N(z)$ related by reflection ρ_p through C_p with $z\in C_p$ as follows:

$$S_N(z) = \left[\frac{1}{z-c_p} + \left(\frac{1}{z-a} - \frac{1}{z-a_p}\right)\right] + \cdots$$
$$\pm\left[\left(\frac{1}{z-a_\nu} - \frac{1}{z-a_{p\nu}}\right) - \left(\frac{1}{z-s_\nu} - \frac{1}{z-s_{p\nu}}\right)\right] + \cdots, \quad (3.49)$$

where the plus sign is used if $|\nu|$ is even and a minus sign if $|\nu|$ is odd. Then, multiplying by $z-c_p$, we have in more detail,

$$
\begin{aligned}
(z-s_p)S_N(z) &= 1+\frac{(z-s_p)/(a-s_p)}{(z-s_p)/(a-s_p)-1}-\frac{(z-s_p)/(a_p-s_p)}{(z-s_p)/(a_p-s_p)-1} \\
&\pm\sum_{j=0}^{N-1}\sum_{i=1}^{m}\sum_{\substack{\nu\in\sigma_j(i),\\ \nu i,\nu_1\neq p}}\left(\frac{(z-s_p)/(a_{\nu i}-s_p)}{(z-s_p)/(a_{\nu i}-s_p)-1}-\frac{(z-s_p)/(\rho_p(a_{\nu i})-s_p)}{(z-s_p)/(\rho_p(a_{\nu i})-s_p)-1}\right) \\
&\pm\sum_{j=0}^{N-1}\sum_{i=1}^{m}\sum_{\substack{\nu\in\sigma_j(i),\\ \nu i,\nu_1\neq p}}\left(\frac{(z-s_p)/(s_{\nu i}-s_p)}{(z-s_p)/(s_{\nu i}-s_p)-1}-\frac{(z-s_p)/(\rho_p(s_{\nu i})-s_p)}{(z-s_p)/(\rho_p(s_{\nu i})-s_p)-1}\right) \\
&\pm(z-s_p)\sum_{\substack{j=1,\\ j\neq p}}^{m}\sum_{i=1}^{m}\sum_{j\nu\in\sigma_N(i)}\left(\frac{a_{j\nu i}-s_{j\nu i}}{(z-a_{j\nu i})(z-s_{j\nu i})}\right).
\end{aligned}
\tag{3.50}
$$

We take the imaginary part of the above expression and using, for instance, $w=(z-s_p)/(a_\nu-s_p)$ and noting that $w^*=(z-s_p)/(\rho_p(a_\nu)-s_p)$, (3.46) gives

$$
\begin{aligned}
&\mathrm{Im}\left\{\frac{(z-s_p)/(a_\nu-s_p)}{(z-s_p)/(a_\nu-s_p)-1}-\frac{(z-s_p)/(\rho_p(z_\nu)-s_p)}{(z-s_p)/(\rho_p(a_\nu)-s_p)-1}\right\} \\
&=\mathrm{Im}\left\{\frac{w}{w-1}-\frac{w^*}{w^*-1}\right\}=0.
\end{aligned}
\tag{3.51}
$$

Taking the imaginary part of (3.50), we see that the first three lines sum to 0. The final $m-1$ terms, all lying inside circles C_i, $i\neq p$, approximate the truncation error and are estimated by

$$\sum_{\nu\in\sigma_{n+1}} r_\nu^2 \leq \Delta^{4N}\sum_{i=1}^{m} r_i^2. \tag{3.52}$$

This gives our final result

$$\mathrm{Im}\{(z-s_p)S_N(z)\} = O(\sqrt{m-1}(\Delta^{2N}(m-1)^{N/2}). \tag{3.53}$$

■

(c) *Circular and radial slit map*

Here we consider the map $w=f(z)$ from the exterior of m discs to the exterior domain bounded by a mixture of radial and circular slits. This map is discussed by Koebe (1916). The mapping formula that we derive here appears to be new. It is not discussed, for instance, in such standard presentations as Nehari (1952) or Schiffer (1950).

Choosing a point $a\in\Omega$, we let $f(a)=0$ and $f(\infty)=\infty$ with $f(z)/z=O(1)$, $z\to\infty$. Reflections through radial slits will keep 0 and ∞ fixed, whereas that through circular slits will swap 0 and ∞ as in the circular slit map above. Let ρ_{ν_e} denote a sequence of reflections with an even number of reflections through

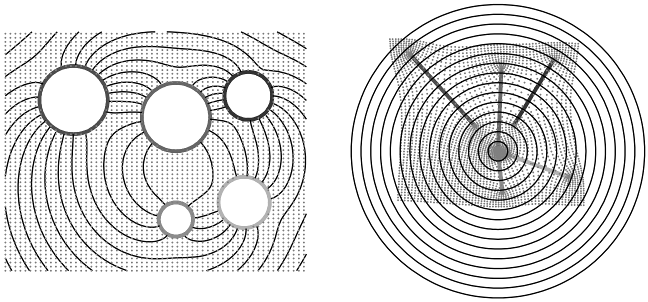

Figure 4. Map to radial slit domain using least-squares method.

circular slits and ρ_{ν_o} denote a sequence with an odd number of reflections through circular slits. Then, $\rho_{\nu_e}(a)$ and $\rho_{\nu_o}(\infty)$ are simple zeros of $f(z)$ and $\rho_{\nu_e}(\infty)$ and $\rho_{\nu_o}(a)$ are simple poles. Therefore, we have

$$f(z) = C(z-a) \prod_{\nu_e,\nu_o} \frac{(z-\rho_{\nu_e}(a))(z-\rho_{\nu_o}(\infty))}{(z-\rho_{\nu_e}(\infty))(z-\rho_{\nu_o}(a))}. \tag{3.54}$$

(Note that the product over $i=1, \dots, m$ is already included in the reflections of a and ∞ and does not appear explicitly here.) Using arguments like those for the radial and circular slit mappings, one can prove that the separation and convergence theorems hold for the mixed radial and circular slit boundary components. We omit the details of the proof. Figure 3 is a graph of an $m=4$ case with two circular and two radial slits produced by evaluating a truncated version of (3.54).

Remark 3.8. Numerical experiments indicate that our convergence criterion for the infinite product formulae is probably not necessary for convergence. We have been unable to find a condition that is both necessary and sufficient.

4. Numerics using least squares

The characterization by means of reflections of the slit maps considered in this paper is natural and leads to straightforward derivations. On the other hand, as the number of circles and slits grows, the required number of reflections for a prescribed accuracy grows exponentially and computation times become impractically large. As an anecdotal example, in one case of the maps like that in figure 4, if m was increased from 3 to 4, the computation time on the third author's laptop increased from 3.5 to 3970 s. Therefore, it is essential to find fast algorithms to compute these maps. We describe such a procedure here.

The idea is closely related to an algorithm given by Trefethen (2005) for finding the Green's function for the exterior of discs. We begin by expressing the desired map f as

$$\log f(z) = \log (z-a) + g(z), \tag{4.1}$$

for a function g that is analytic in Ω (and its boundary, according to equations (3.1) and (3.35)). This form imposes the normalizations $f(a)=0$ and $f(\infty)=\infty$.

Box 1.
MATLAB code for finding the parameters and computing the values of a radial slit map

```
function f = radialslitmap(center,radius,N,J)

m = length(center);

  % Returns the inverse powers used as a basis for the solution.
  function A = basisfuncs(z)
    for k = 1:m
      for j=1:J
        A(:,J*(k-1)+j) = 1./(z-center(k)).^j;
      end
    end
  end

% Points on circles.
z = zeros(N,m);
for k = 1:m
  z(:,k) = center(k) + radius(k)*exp(2i*pi*(0:N-1)'/N);
end
z = z(:);

% Basis functions
A = basisfuncs(z);

% Convert to real form.
ARI = [ imag(A), real(A) ];
% Pairwise differences around each individual circle.
D = toeplitz( [1 -1 zeros(1,N-2)], [ 1 zeros(1,N-2) -1] );
E = kron(eye(m),D);

% Set up and solve the linear least-squares system
rhs = -E*imag(log(z));
xRI = (E*ARI) \ rhs;
x = xRI(1:end/2) + 1i*xRI(end/2+1:end);

fprintf(' residual estimate = %.3e\n\n',norm(E*ARI*xRI - rhs))

% Create a callable function for the map
f = @rsmap;

  function w = rsmap(z)
    A = basisfuncs(z(:));
    w = zeros(size(z));
    w(:) = log(z(:)) + A*x;
  end

end
```

Box 2.
Driver for the code in box 1 for the map from an exterior domain bounded by five circles to a radial slit domain

```
% Circle parameters.
center = [3*exp(1i*pi/4), -1.5i, -3+2i, 2-1i, 1.5i];
radius = [ 0.7 0.5 1 .75 1];
m = length(center);

% Discretization parameters.
N = 300;  J = 20;

% Solve for the map.
f = radialslitmap(center,radius,N,J);

% Points on the circles.
zc = repmat(exp(2*pi*1i*(0:200)'/200),1,m)*diag(radius);
zc = repmat(center,201,1) + zc;

% Define grid of points in the circle domain.
x = linspace(min(real(zc(:)))-1,max(real(zc(:)))+1,120);
y = linspace(min(imag(zc(:)))-1,max(imag(zc(:)))+1,120);
[X,Y] = meshgrid(x,y);  Z = X + 1i*Y;
% Keep only those points outside the circles.
mask = true(size(Z));
for k = 1:m
  mask(abs(Z-center(k))<radius(k)) = false;
end
Z(~mask) = NaN;

W = Z;  W(mask) = f(Z(mask));   % evaluation on the grid

% Plotting of level curves and point images.
subplot(1,2,1)    % circle domain
plot(zc,'.-')
axis equal, hold on
levels = ((1:15)/16) * exp( max(real(W(:))) );
contour(X,Y,real(W),log(levels),'k')
plot(Z(1:2:end,1:2:end),'.','color',[.5 .5 .5],'markersize',3)
subplot(1,2,2)    % slit domain
plot(exp(f(zc)),'.-')
axis equal, hold on
plot(exp(W(1:2:end,1:2:end)),'.','color',[.5 .5 .5],'markersize',3)
plot(exp(2*pi*1i*(0:200)'/200)*levels,'k')
```

The remainder $g(z)$ is then expanded in the form

$$g(z) \approx \sum_{k=1}^{m} \sum_{j=1}^{J} \frac{\alpha_{k,j}}{(z - c_k)^j}, \tag{4.2}$$

which allows singularities in each of the circles. In practice, we discretize the boundary of Ω by placing N equally spaced points on each of the circles and express the double sum of (4.2) as a matrix–vector product Ax, where each column of the matrix A is the discretization of some $(z-c_k)^{-j}$ and $x=[\alpha_{k,j}]$.

The unknown coefficients in x are determined by the fact that $\mathrm{Im}\,(\log f)$ is constant on radial slits and $\mathrm{Re}\,(\log f)$ is constant on circular slits. Indeed, the key fact is that we can impose these conditions *linearly*. To do this, we need to break both A and x into its real and imaginary parts. Letting $A=A_\mathrm{R}+\mathrm{i}A_\mathrm{I}$ and $x=x_\mathrm{R}+\mathrm{i}x_\mathrm{I}$, we trivially get

$$[\mathrm{Re}\, g] = A_\mathrm{R}x_\mathrm{R} - A_\mathrm{I}x_\mathrm{I}, \qquad [\mathrm{Im}\, g] = A_\mathrm{I}x_\mathrm{R} + A_\mathrm{R}x_\mathrm{I}. \tag{4.3}$$

For concreteness, let us continue the discussion in terms of the radial slit case. The constant values of $\mathrm{Im} \log f$ on each slit are not known in advance. Instead, we ask that pairwise differences of $\mathrm{Im} \log f$ be zero around each circle. Defining

$$D = \begin{bmatrix} 1 & & & & -1 \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}_{N\times N}, \qquad E = \begin{bmatrix} D & & & \\ & D & & \\ & & \ddots & \\ & & & D \end{bmatrix}_{mN\times mN}, \tag{4.4}$$

we arrive at the expression

$$E[\,A_I \quad A_R\,]\begin{bmatrix} x_R \\ x_I \end{bmatrix} \approx -E[\mathrm{Im}\,\log(z-a)], \tag{4.5}$$

which is an ordinary linear least-squares problem for the unknown coefficients. This problem can be solved very quickly even for fairly large discretizations.

Box 1 shows a MATLAB code based on these ideas. The expression (4.1) and (4.2) for the map is so simple that the function returns a callable object that evaluates to the computed function. Box 2 illustrates how the code can be used to map points and create level curves for a domain bounded by five circles. This example is given in figure 4. Computing the map parameters (setting up and solving the least-squares system) took approximately 3 s on a 1.4 GHz Pentium-M laptop.

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