SCHWARZ–CHRISTOFFEL MAPPING OF MULTIPLY CONNECTED DOMAINS

By

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Abstract. A Schwarz–Christoffel mapping formula is established for polygonal domains of finite connectivity $m \geq 2$ thereby extending the results of Christoffel (1867) and Schwarz (1869) for $m = 1$ and Komatu (1945), $m = 2$. A formula for f, the conformal map of the exterior of m bounded disks to the exterior of m bounded disjoint polygons, is derived. The derivation characterizes the global preSchwarzian $f''(z)/f'(z)$ on the Riemann sphere in terms of its singularities on the sphere and its values on the m boundary circles via the reflection principle and then identifies a singularity function with the same boundary behavior. The singularity function is constructed by a "method of images" infinite sequence of iterations of reflecting prevertex singularities from the m boundary circles to the whole sphere.

1 Introduction

Recently, it was remarked that "It is a longstanding dream to generalize [Schwarz–Christoffel] to multiply connected domains...", [5], p. 7. In this paper, we carry out that generalization for domains of connectivity $m \geq 2$ by deriving a formula for the conformal map of an m-connected, unbounded circular domain Ω onto an unbounded polygonal domain $\mathbb{P}$. The boundary of $\mathbb{P}$ consists of m disjoint polygonal curves. The point at infinity is in the interior of both domains and is held fixed by the mapping $f : \Omega \to \mathbb{P}$, $f(\infty) = \infty$. The boundary components of $\mathbb{P}$ may be polygonal Jordan curves, simple polygonal "two-sided" slits, or polygonal Jordan curves with some attached simple polygonal "two-sided" slits protruding into $\mathbb{P}$. Our construction of the mapping function is based on infinite sequences of iterated reflections that generate infinite products in our mapping formula

$$f(z) = \int^{z} \prod_{i=1}^{m} \prod_{k=1}^{K_i} \left[\prod_{\substack{j=0 \\ \nu \in \sigma_j(i)}}^{\infty} \left(\frac{\zeta - z_{k,\nu i}}{\zeta - s_{\nu i}} \right) \right]^{\beta_{k,i}} d\zeta. \tag{1.1}$$

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 94 (2004)

The $z_{k,\nu j}$ are generated by reflections of the prevertex points on $\partial\Omega$. The $s_{\nu j}$ are generated by the point at infinity and its reflections, and the $\beta_{k,j}\pi$ are the turning angles at the vertices of $\partial\mathbb{P}$. The ν's are multiindices. We explain all of these matters below. Our proof of the convergence of the infinite products in (1.1), and hence the validity of the formula, for connectivity $m \geq 3$ is valid for domains whose boundary components are disjoint and satisfy an additional separation condition described in the proof and discussed further in the paper. It should be possible to prove that disjointness is sufficient for (1.1). Our technique of using an infinite sequence of iterated reflections and analytic continuation by reflection to construct the mapping formula is related to the "method of images" used in electrostatics; cf. [1].

The problem of finding a conformal map from a simple type of canonical domain to a general domain has a long history and has led to much elegant mathematical theory. If the domains are simply connected, the canonical domain is usually taken to be the unit disk and the theory is that associated with the Riemann Mapping Theorem. For multiply connected domains, various choices of "canonical" domain have been identified and studied [9], [8], [17]. The one we have chosen is the exterior of a finite number of nonoverlapping disks, which seems especially appropriate with the target domain being an exterior domain. Furthermore, there are various classical explicit formulas relating pairs of some of the canonical classes of slit domains; but the circular domains seem to stand apart by themselves in this respect. However, since the parallel slit canonical domains are unbounded polygonal domains, our S–C mapping formula provides an explicit formula connecting mappings in the circular class with those in the parallel slit class of canonical domains. Also, the circular boundary components are convenient for the use of Fourier analysis of boundary data in applied problems. For a general target domain, there is no explicit mapping formula. On the other hand, in the simply connected case, if the target is bounded by a polygon, then the Schwarz–Christoffel (S–C) formula (3.1) gives an explicit representation of the map in terms of quadratures and a finite number of parameters (which must be determined numerically [6]).

The (S–C) formula for the conformal map of an annulus onto a doubly connected polygonal domain (3.2) has been known for more than fifty years [12]; cf. [3]. However, with the exception of special configurations involving much symmetry allowing reduction to a simply connected mapping problem, [7], [15], no explicit generalization to connectivity $m \geq 3$ seems to be known; see also [13], [14].

In our recent paper [3], we gave a new derivation of the annulus formula based on the characterization of the global preSchwarzian $f''(z)/f'(z)$ of the mapping function in terms of its singularities and boundary behavior, a construction of a global singularity function $S(z)$ for the analytic continuation of the preSchwarzian, and then a proof that $f''(z)/f'(z) = S(z)$. The construction consists of an infinite sequence of repeated reflections which generate infinite products and the theta functions in the final formula for the mapping function (3.2). Our present work for higher connectivity uses the same general strategy, but the step to connectivity greater than two introduces new and substantial difficulties. The principal new feature that distinguishes the case $m \geq 3$ from $m = 1$ or 2 is an exponential increase of terms by a factor of $m-1$ at each of the infinitely many levels of reflections in the construction of the singularity function and the attendant difficulties in describing and establishing the properties of $S(z)$.

The remainder of the paper is organized as follows: In Section 2, we set notation and present some supporting results on reflections, moduli, separation and estimates of radii. The global preSchwarzian, its singularity function, the main result (Theorem 1) and its corollaries for connectivity $m = 2$ are in Section 3. Convergence of the limit defining $S(z)$ for a region whose separation modulus Δ satisfies the bound $\Delta < 1/(m-1)^{1/4}$ is established in Section 4. In Section 5 we show that $S(z)$ satisfies the boundary condition necessary for proving that $S(z) = f''(z)/f'(z)$. Some elementary graphical illustrations are given in Section 6.

2 Preliminaries

Throughout this work, $\mathbb{C}$ is the complex plane and $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. We let Ω denote an m-connected unbounded circular domain containing the point at infinity which is conformally equivalent to the unbounded polygonal domain $\mathbb{P}$. That is, $\partial\Omega$ consists of m disjoint circles $C_j = \{z : |z - c_j| = r_j\}$, $j = 1, \ldots, m$, and $\Omega = \mathbb{C}_\infty \setminus \bigcup_{j=1}^{m} clD_j$ where $clD_j = \{z : |z - c_j| \leq r_j\}$ are the m disks with mutually disjoint closures.The corresponding polygonal boundary components $f(C_j)$ of $\mathbb{P}$ are denoted by Γ_j, $j = 1, \ldots, m$ and given the counterclockwise orientation. The K_j vertices of Γ_j are denoted by $w_{k,j}, k = 1, \ldots, K_j$ numbered counterclockwise around Γ_j. The corresponding vertex angles of the Γ_j at the vertices $w_{k,j}$, measured from the interior of $\mathbb{P}$, are denoted by $\pi\alpha_{k,j}$, where $0 < \alpha_{k,j} \leq 2$, and $\beta_{k,j}\pi$ is the turning of the tangent at $w_{j,k}$ where $\beta_{k,j} = \alpha_{k,j} - 1$. The *prevertices* are denoted by $z_{k,j} \in C_j$ with $f(z_{k,j}) = w_{k,j}$.

2.1 Reflections. Throughout this work, the *reflection* of z through a circle C with center c and radius r is given by

$$z^* = \rho_C(z) := c + \frac{r^2}{\overline{z} - \overline{c}},$$

i.e., z and z^* are *symmetric points with respect to the circle* C. If $C = C_\tau$, where τ is an index of a circle, we denote ρ_{C_τ} by ρ_τ.

In order to explain the reflection process and the indexing of reflections of boundary circles, prevertices and the centers s_j in our work, we first describe the initial steps in the triply connected case which anticipate the general situation when the connectivity exceeds three. We begin with the unbounded, triply connected circular domain Ω, $\partial\Omega = C_1 \cup C_2 \cup C_3$, which f maps conformally onto an unbounded triply connected polygonal domain $\mathbb{P}$. On each circle C_j, there are K_j prevertices $z_{k,j}$, $k = 1, ..., K_j$, $j = 1, 2, 3$. The reflection process begins by reflecting Ω through C_1 to form $\Omega_1 := \rho_1(\Omega)$, reflecting Ω through C_2 to obtain $\Omega_2 := \rho_2(\Omega)$ and Ω through C_3 to obtain $\Omega_3 := \rho_3(\Omega)$. Each of the three domains Ω_j is a bounded triply connected circular domain with circular boundary components C_j, C_{ji}, $i \neq j$, $1 \leq i \leq 3$:

$$\partial\Omega_1 = C_1 \cup C_{12} \cup C_{13}, \quad \partial\Omega_2 = C_2 \cup C_{21} \cup C_{23}, \quad \partial\Omega_3 = C_3 \cup C_{31} \cup C_{32}.$$

At the next level, there are six reflections: Ω_1 is reflected through its interior boundary circles C_{12} and C_{13} and similarly for Ω_2 and Ω_3 producing the six bounded, triply connected circular domains

$$\begin{aligned} &\Omega_{12:=}\rho_{12}(\Omega_1), \quad \Omega_{13:=}\rho_{13}(\Omega_1), \quad \Omega_{21:=}\rho_{21}(\Omega_2), \quad \Omega_{23:=}\rho_{23}(\Omega_2), \\ &\Omega_{31:=}\rho_{31}(\Omega_3), \quad \Omega_{32:=}\rho_{32}(\Omega_3). \end{aligned}$$

The new boundary circles are

$$\begin{aligned} &C_{121} = \rho_{12}(C_1), \quad C_{123} = \rho_{12}(C_{13}), \quad \partial\Omega_{12} = C_{12} \cup C_{121} \cup C_{123}, \\ &C_{131} = \rho_{13}(C_1), \quad C_{132} = \rho_{13}(C_{12}), \quad \partial\Omega_{13} = C_{13} \cup C_{131} \cup C_{132} \end{aligned}$$

with similar results and notation for the boundaries of Ω_{21}, Ω_{23}, Ω_{31},and Ω_{32}. One sees that by continuing in this fashion the number of new regions and new boundary components created by the reflections at a given level is twice that at the preceding level; see Figure 1. Furthermore, the general m-connected case is similar, with $m - 1$ replacing the factor 2 in the exponential rate of increase of regions and boundary circles.

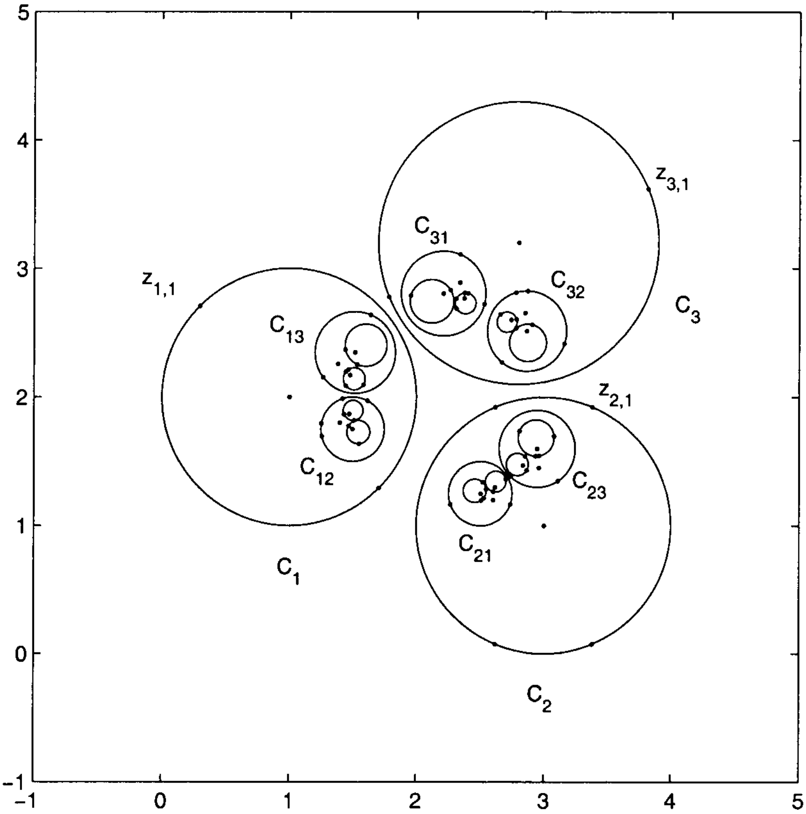

Figure 1. Circle reflection notation.

We now describe the iterated reflection process for the general m-connected unbounded circular domain Ω with boundary components $C_1, ..., C_m$. Fixing an index $\nu_1 \in \{1, ..., m\}$, we reflect Ω through the circle C_{ν_1} to form an m-connected bounded circular region Ω_{ν_1} with outer boundary C_{ν_1} and $m-1$ holes bounded by circles $C_{\nu_1 j}$,

$$\Omega_{\nu_1} := \rho_{\nu_1}(\Omega), \quad C_{\nu_1 j} = \rho_{\nu_1}(C_j), \quad j \neq \nu_1, j = 1, ..., m.$$

Similarly, reflecting Ω through each of the other $m-1$ boundary circles, $C_k, k \neq \nu_1$, produces circular domains with outer boundary C_k, $m-1$ holes and inner boundary circles

$$\Omega_k := \rho_k(\Omega), \quad C_{kj} := \rho_k(C_j), \quad j \neq k, \quad j = 1, ..., m.$$

This set of m reflections and corresponding circular subdomains, Ω_k, of the complement is the first step in the sequence of iterations of reflections of domains through boundary circles which leads to an exhaustion of the complement of Ω except for a set of limit points.

At the next level, we produce $m(m-1)$ domains Ω_{kj} by reflecting each Ω_k domain through each of its $m-1$ interior boundary circles C_{kj}. For each k, this produces the $m-1$ circular domains $\Omega_{kj} := \rho_{kj}(\Omega_k)$ $(j \neq k, j = 1, ..., m)$ of connectivity m, each with outer boundary C_{kj}, and the $m-1$ inner boundary circles

$$C_{kjk} := \rho_{kj}(C_k), \quad C_{kji} = \rho_{kj}(C_{ki}), \quad i \in \{1, ..., m\} \setminus \{j, k\}.$$

We note that for each fixed k and $j \neq k$, the first circle C_{kjk} is the reflection of the outer boundary of Ω_k, while each of the other $m-2$ circles is the reflection of one of the inner boundary components of Ω_k.

In general, the reflected regions and circles are labeled with multi-indices $\nu = \nu_1\nu_2 \cdots \nu_n$ with $\nu_j \in \{1, ..., m\}$, $\nu_k \neq \nu_{k+1}$, $k = 1, ..., n-1$; we write $|\nu|$ to denote the *length of* ν, i.e., $|\nu| = n$.

Definition 1. The set of multi-indices of length n is denoted

$$\sigma_n = \{\nu_1\nu_2 \cdots \nu_n : 1 \leq \nu_j \leq m, \nu_k \neq \nu_{k+1}, k = 1, ..., n-1\}, \quad n > 0,$$

and $\sigma_0 = \phi$, in which case $\nu i = i$. Also,

$$\sigma_n(i) = \{\nu \in \sigma_n : \nu_n \neq i\}$$

denotes sequences in σ_n whose last term never equals i.

Thus, if $\nu \in \sigma_n$, $n > 1$, $\Omega_\nu = \rho_\nu\left(\Omega_{\nu_1 \cdots \nu_{n-1}}\right)$ is a circular domain with outer boundary C_ν and $m-1$ interior boundary circles

(2.1)
$$C_{\nu\nu_{n-1}} = \rho_\nu\left(C_{\nu_1 \cdots \nu_{n-1}}\right), \quad C_{\nu j} = \rho_\nu\left(C_{\nu_1 \cdots \nu_{n-1} j}\right), \quad j \in \{1, \ldots, m\} \setminus \{\nu_{n-1}, \nu_n\}.$$

Note that if $\nu \in \sigma_n$ then for $j \neq \nu_n$, $\Omega_{\nu j} = \rho_{\nu j}(\Omega_\nu)$. Clearly σ_n contains $m(m-1)^{n-1}$ elements, which is consistent with our earlier comment that the number of circular domains Ω_ν at a particular level of reflections, say $\nu \in \sigma_n$, is $m-1$ times the number of domains $\Omega_{\widetilde{\nu}}$, $\widetilde{\nu} \in \sigma_{n-1}$, at the preceding level.

It will be necessary to follow the successive reflections of $c_1, ..., c_m$, the centers of the boundary circles $C_1, ..., C_m$. Each c_k is the center of the circular domain Ω_k because it is the reflection of ∞ through C_k. Clearly, none of the successive iterated reflections of c_k will be the centers of the corresponding reflected domains, Ω_ν, $\nu \in \sigma_n$. However, we can index each of these reflected points with the index of the reflected domain in which it lies, i.e., $s_\nu = \rho_\nu\left(s_{\nu_1 \cdots \nu_{n-1}}\right)$, $\nu \in \sigma_n$, and $s_k := c_k$,

$k = 1, \ldots, m$ (to avoid confusion with the "center" c_ν of the reflected circular domain). We also note that if $\nu \in \sigma_n$, then for $j \neq \nu_n$, $s_{\nu j} = \rho_{\nu j}(s_\nu)$.

Clearly, one should index the reflections of the prevertices, $z_{k,j}$, with the indices of the reflected circles on which they lie. Thus, if $\nu \in \sigma_n$, then $z_{k,\nu}$ is the (iterated) reflection of z_{k,ν_1} that lies on the circle C_ν, and from (2.1)

$$z_{k,\nu\nu_{n-1}} = \rho_\nu\left(z_{k,\nu_1\cdots\nu_{n-1}}\right), \; z_{k,\nu j} = \rho_\nu\left(z_{k,\nu_1\cdots\nu_{n-1}j}\right), \; j \in \{1,\ldots,m\} \setminus \{\nu_{n-1}, \nu_n\}.$$

Similarly, we let r_ν denote the radius of a circle C_ν, $\nu \in \sigma_n$.

The following elementary fact about successive reflections will be useful.

Proposition 1. *Reflection of a set of points U through circle C_λ followed by reflection through circle C_τ is the same as its reflection through circle C_τ followed by reflection through $C_{\tau\lambda} = \rho_\tau(C_\lambda)$, the reflection of C_λ through C_τ. Symbolically,*

$$\rho_\tau(\rho_\lambda(u)) = \rho_{\tau\lambda}(\rho_\tau(u)).$$

Proof. For simplicity in visualizing the geometry, one may think of the case when the two circles are nonintersecting. Since Moebius transformations preserve symmetry (reflection) in circles it is enough to assume that C_τ is the real axis. Thus $\rho_\tau(u) = \overline{u}$, the circles C_λ and $C_{\tau\lambda}$ are reflections of each other in the real axis, $c_{\tau\lambda} = \overline{c_\lambda}$, $r_{\tau\lambda} = r_\lambda$ and

$$\rho_\tau(\rho_\lambda(u)) = \overline{c_\lambda + \frac{r_\lambda^2}{\overline{u} - \overline{c_\lambda}}} = c_{\tau\lambda} + \frac{r_{\tau\lambda}^2}{\overline{\rho_\tau(u)} - \overline{c_{\tau\lambda}}} = \rho_{\tau\lambda}(\rho_\tau(u)). \qquad \square$$

The next result is used in Section 5.

Lemma 1. *Let $\nu = \nu_1 \cdots \nu_n \in \sigma_n$ and $j \in \{1, \ldots, m\} \setminus \{\nu_1\}$. Then*

$$C_{j\nu} = \rho_j(C_\nu), \quad s_{j\nu} = \rho_j(s_\nu) \quad \textit{and} \quad z_{k,j\nu} = \rho_j(z_{k,\nu}).$$

Proof. We first show that $C_{j\nu} = \rho_j(C_\nu)$. This follows from Proposition 1 and is needed to apply Proposition 1 when $\tau = j$ and $\lambda = \nu$. Our proof is by induction on $|\nu|$.

For $|\nu| = 1, \nu = \nu_1 \neq j$, we have $C_{j\nu_1} = \rho_j(C_{\nu_1})$ by the definition of $C_{j\nu_1}$ in (2.1).

For $|\nu| = 2, \nu = \nu_1\nu_2, \nu_1 \neq j$, there are two cases. First, for $\nu_2 \neq j$, we have

$$\begin{aligned} C_{j\nu_1\nu_2} &= \rho_{j\nu_1}(C_{j\nu_2}) \text{ by (2.1)} \\ &= \rho_{j\nu_1}(\rho_j(C_{\nu_2})) \text{ by definition of } C_{j\nu_2} \\ &= \rho_j(\rho_{\nu_1}(C_{\nu_2})) \text{ by Proposition 1 with } \tau = j,\ \lambda = \nu_1 \\ &= \rho_j(C_{\nu_1\nu_2}) \text{ by definition of } C_{\nu_1\nu_2}. \end{aligned}$$

Next, for $\nu_2 = j$, we have

$$\begin{aligned} C_{j\nu_1 j} &= \rho_{j\nu_1}(C_j) \text{ by (2.1)} \\ &= \rho_{j\nu_1}(\rho_j(C_j)) \text{ since } \rho_j(C_j) = C_j \\ &= \rho_j(\rho_{\nu_1}(C_j)) \text{ by Prop. 1} \\ &= \rho_j(C_{\nu_1 j}) \text{ by definition of } C_{\nu_1 j}. \end{aligned}$$

We now use our induction hypothesis, $C_{j\nu} = \rho_j(C_\nu)$ for $|\nu| < n$ and show that the result is true for $|\nu| = n > 2$. Thus, let $\nu = \nu_1\nu_2 \cdots \nu_n, \nu_1 \neq j$. First, if $\nu_n \neq \nu_{n-2}$ we have

$$\begin{aligned} C_{j\nu} &= \rho_{j\nu_1\cdots\nu_{n-1}}(C_{j\nu_1\cdots\nu_{n-2}\nu_n}) \text{ by (2.1)} \\ &= \rho_{j\nu_1\cdots\nu_{n-1}}(\rho_j(C_{\nu_1\cdots\nu_{n-2}\nu_n})) \text{ by the induction hypothesis} \\ &= \rho_j(\rho_{\nu_1\cdots\nu_{n-1}}(C_{\nu_1\cdots\nu_{n-2}\nu_n})) \text{ by Prop. 1 and the induction hypothesis} \\ &= \rho_j(C_\nu) \text{ by (2.1).} \end{aligned}$$

When $\nu_n = \nu_{n-2}$, we have

$$\begin{aligned} C_{j\nu} &= \rho_{j\nu_1\cdots\nu_{n-1}}(C_{j\nu_1\cdots\nu_{n-2}}) \text{ by (2.1)} \\ &= \rho_{j\nu_1\cdots\nu_{n-1}}(\rho_j(C_{\nu_1\cdots\nu_{n-2}})) \text{ by the induction hypothesis} \\ &= \rho_j(\rho_{\nu_1\cdots\nu_{n-1}}(C_{\nu_1\cdots\nu_{n-2}})) \text{ by Prop. 1 and the induction hypothesis} \\ &= \rho_j(C_{j\nu_1\cdots\nu_{n-1}\nu_{n-2}}) \text{ by (2.1)} \\ &= \rho_j(C_\nu) \text{ since } \nu_{n-2} = \nu_n. \end{aligned}$$

Similar reasoning yields the result $z_{k,j\nu} = \rho_j\left(z_{k,\nu}\right)$, since $z_{k,j\nu} \in C_{j\nu}$.

We now note that $s_{\nu q} = \rho_{\nu q}\left(s_\nu\right)$ $q \neq \nu_n$(by the definition of $s_{\nu q}$) and then argue inductively. Recalling that $s_j = \rho_j\left(\infty\right)$, $s_{\nu_1} = \rho_{\nu_1}\left(\infty\right)$ and applying Proposition 1, we find that

$$s_{j\nu_1} = \rho_{j\nu_1}\left(s_j\right) = \rho_{j\nu_1}\left(\rho_j\left(\infty\right)\right) = \rho_j\left(\rho_{\nu_1}\left(\infty\right)\right) = \rho_j\left(s_{\nu_1}\right)$$

and similarly $s_{j\nu_1\nu_2} = \rho_{j\nu_1\nu_2}\left(s_{j\nu_1}\right) = \rho_{j\nu_1\nu_2}\left(\rho_j\left(s_{\nu_1}\right)\right) = \rho_j\left(\rho_{\nu_1\nu_2}\left(s_{\nu_1}\right)\right) = \rho_j\left(s_{\nu_1\nu_2}\right)$. Using the induction assumption $s_{j\nu_1\cdots\nu_{n-1}} = \rho_j\left(s_{\nu_1\cdots\nu_{n-1}}\right)$, the first part of the lemma, and Proposition 1, we argue that

$$s_{j\nu} = \rho_{j\nu}\left(s_{j\nu_1\cdots\nu_{n-1}}\right) = \rho_{j\nu}\left(\rho_j\left(s_{\nu_1\cdots\nu_{n-1}}\right)\right) = \rho_j\left(\rho_\nu\left(s_{\nu_1\cdots\nu_{n-1}}\right)\right) = \rho_j\left(s_\nu\right). \qquad \square$$

2.2 Moduli and separation. Let μ_{jk}^{-1} denote the conformal modulus of the doubly connected region bounded by the curves Γ_j and Γ_k. That is, the region is conformally equivalent to the annulus $\mu_{jk} < |z| < 1$. For the region exterior to two mutually exterior disks with centers c_j, c_k, radii r_j, r_k and distance between centers $d_{j,k} = |c_j - c_k|$, one has the elementary formula

$$\mu_{jk} = \frac{d_{j,k}^2 - r_j^2 - r_k^2 - \sqrt{\left[(d_{j,k} - r_j)^2 - r_k^2\right]\left[(d_{j,k} + r_j)^2 - r_k^2\right]}}{2r_j r_k}, \tag{2.2}$$

for the conformal invariant μ_{jk} obtained with the Moebius mapping of the region onto the annulus $\mu_{jk} < |z| < 1$.

The following lemmas are used in our convergence proofs. We denote the area enclosed by a curve Γ by $\alpha(\Gamma)$.

Lemma 2. *Let B be a bounded doubly connected region with finite modulus $\mu_{01}^{-1} > 1$, bounded on the outside by a Jordan curve Γ_0 and on the inside by a Jordan curve Γ_1. Then*

$$\alpha(\Gamma_1) \leq \mu_{01}^2 \alpha(\Gamma_0).$$

Proof. See [10], Lemma 17.7c (a), p. 503. □

Note that if Γ_0 and Γ_1 are two circles of radii r_0 and r_1, then the lemma is $r_1 \leq \mu_{01} r_0$.

It is essential to have the disjointness (separation) of the m boundary circles C_j expressed analytically. We do this with the *nonoverlap inequalities*

$$\mu_{jk}^{sep} := \frac{r_j + r_k}{|c_j - c_k|} = \frac{r_j + r_k}{d_{j,k}} < 1, \quad j \neq k,\ 1 \leq j, k \leq m. \tag{2.3}$$

We now define the *separation modulus* of the region

$$\Delta := \max_{i,j;i \neq j} \mu_{ij}^{sep} \tag{2.4}$$

for the m boundary circles C_j, cf. [10], p. 501. Let $\tilde{C}_j$ denote the circle with center c_j and radius r_j/Δ; then geometrically, $1/\Delta$ is the smallest magnification of the m radii such that at least two $\tilde{C}_j$'s just touch.

The quantities μ_{jk}^{sep} in (2.3) are not conformally invariant, but one can express the nonoverlapping property in terms of the conformally invariant quantities $\alpha_{j,k}$ defined by

$$\alpha_{j,k} := \frac{1}{2}\left(\mu_{jk} + \frac{1}{\mu_{jk}}\right) = \frac{d_{j,k}^2 - \left(r_j^2 + r_k^2\right)}{2r_j r_k}. \tag{2.5}$$

The second equality follows from (2.2). Two mutually exterior circles C_j and C_k are nonoverlapping if and only if $d_{j,k}^2 > (r_j + r_k)^2$, i.e., if and only if $\alpha_{j,k} > 1$.

Proposition 2. $\mu_{jk} < \left(\mu_{jk}^{sep}\right)^2 \leq \Delta^2$.

Proof. To simplify notation, we consider a typical pair of circles C_j, C_k and write $\mu_s = \mu_{j,k}^{sep}$, $\alpha = \alpha_{j,k}$, $d = d_{j,k}$. Thus $(\mu_s d)^2 = (r_j + r_k)^2$, and from (2.5), $\mu_s^2 \left(r_j^2 + r_k^2 + 2\alpha r_j r_k\right) = r_j^2 + r_k^2 + 2r_j r_k$. Further manipulation and the geometric-arithmetic mean inequality give

$$\alpha\mu_s^2 - 1 = \left(1 - \mu_s^2\right) \frac{r_j^2 + r_k^2}{2r_j r_k} \geq 1 - \mu_s^2$$

with equality if and only if $r_j = r_k$. Continuing, we find that

$$\frac{1}{\mu_s^2} \leq \frac{\alpha + 1}{2} = \left[\frac{1}{2}\left(\sqrt{\mu_{jk}} + \frac{1}{\sqrt{\mu_{jk}}}\right)\right]^2,$$

and hence $\sqrt{\mu_{jk}} < \mu_s$, since $\sqrt{\mu_{jk}} < 2\sqrt{\mu_{jk}} / \left(\mu_{jk} + 1\right) \leq \mu_s$. The inequality is strict since $\mu_{jk} < 1$. Clearly, for many nonoverlapping circles, the result follows since $\Delta = \max_{j,k;j\neq k} \mu_{jk}^{sep}$. □

Lemma 3 ([10], p. 505).

$$\sum_{\nu\in\sigma_{n+1}} \alpha(\tilde{C}_\nu) \leq \Delta^{4n} \sum_{i=1}^{m} \alpha(\tilde{C}_i),$$

$$\sum_{\nu\in\sigma_{n+1}} r_\nu^2 \leq \Delta^{4n} \sum_{i=1}^{m} r_i^2.$$

Proof. (*Idea of proof* for $m = 3$.) Here $(\tilde{C}_j)_\nu$ denotes the reflection of $\tilde{C}_j$ through C_ν. (This is not the magnification of C_ν by $1/\Delta$ unless the circles are concentric.) By Proposition 2, Lemma 2 and the geometry of the areas, $\alpha(C_\nu) \leq \Delta^2 \alpha(\tilde{C}_\nu)$. Thus, for instance, we have

$$\alpha(\tilde{C}_{13}) + \alpha(\tilde{C}_{12}) \leq \Delta^2 \alpha(C_1) \leq \Delta^4 \alpha(\tilde{C}_1),$$

$$\alpha(\tilde{C}_{23}) + \alpha(\tilde{C}_{21}) \leq \Delta^2 \alpha(C_2) \leq \Delta^4 \alpha(\tilde{C}_2),$$

$$\alpha(\tilde{C}_{32}) + \alpha(\tilde{C}_{31}) \leq \Delta^2 \alpha(C_3) \leq \Delta^4 \alpha(\tilde{C}_3).$$

This gives the first step, and $\alpha(C_\nu) = \pi r_\nu^2$ gives the second inequality. □

3 Analytic continuation and the mapping formula

In this section, we describe the analytic continuation by reflection of the mapping function, the singularities of the *preSchwarzian* $f''(z)/f'(z)$ and its singularity function $S(z)$. We give a precise statement of our mapping formula and its proof in Theorem 1. The convergence and boundary property of $S(z)$ are used here, but the somewhat cumbersome details of these proofs are postponed to the following two sections of the paper.

We begin with some general observations about Schwarz–Christoffel type mapping formulas of the unit disk and the annulus that motivate our present work for the m-connected circular domain. In each case, the formula is an integral of a product function

$$f(z) = \int^{z} P(\zeta)\, d\zeta,$$

or equivalently $f'(z) = P(z)$. For mapping the unit disk to a polygonal domain, the geometry of the polygon determines that $\arg(f'(e^{it})\, ie^{it})$ is constant on arcs of $|z| = 1$ with jumps $\beta_k \pi$ at the prevertices, z_k, and hence the product has the form

$$f'(z) = P(z) = \prod_{k=1}^{K} (z - z_k)^{-\beta_k}. \tag{3.1}$$

Mapping the annulus $\mu < |z| < 1$ onto a conformally equivalent doubly connected polygonal domain leads to infinite products

$$f'(z) = P(z) = \prod_{k=1}^{K} \left[\Theta\left(\frac{z}{\mu z_{0,k}}\right)\right]^{-\beta_{0,k}} \prod_{j=1}^{J} \left[\Theta\left(\frac{\mu z}{z_{1,j}}\right)\right]^{-\beta_{1,j}} \tag{3.2}$$

involving the theta function, $\Theta(w) = \prod_{\nu=0}^{\infty} \left(1 - \mu^{2\nu+1} w\right)\left(1 - \mu^{2\nu+1}/w\right)$, [12], [3]. Although the local analysis of the boundary mapping of the annulus is driven by the same geometric idea as for (3.1), it is not obvious why there are infinite products and what they should be. One approach to this problem is to use the geometric properties of the mapping function f under reflections and the affine invariance of the preSchwarzian to generate a formula for the analytic continuation of a globally defined, single-valued preSchwarzian [3]. Note that in terms of the product function, one has $f''(z)/f'(z) = P'(z)/P(z)$. We now use this general strategy for the m-connected domains in the present work.

Our derivation of the mapping formula (1.1) begins by considering the analytic continuation of f from Ω to a domain Ω_j by reflection across an arc $\gamma_{k,j}$ between successive prevertices $z_{k,j}, z_{k+1,j}$ on C_j. Such an extension $\tilde{f}_{k,j}$ has the form

$$\tilde{f}_{k,j}(z) = f(z), \quad z \in \Omega \cup \gamma_k, \quad \tilde{f}_{k,j}(z) = a_{k,j}\overline{f(c_j + r_j^2/(\overline{z} - \overline{c_j}))} + b_{k,j}, \quad z \in \Omega_j,$$

with $a_{k,j}, b_{k,j}$ determined by the line containing the edge $f(\gamma_{k,j})$ joining $w_{k,j}$ and $w_{k+1,j}$ in the boundary of $\mathbb{P}$. This reflection maps Ω_j conformally onto $\mathbb{P}_{k,j}$, the unbounded, m-connected polygonal domain obtained by reflecting $\mathbb{P}$ across the line containing the segment $f(\gamma_k)$. In general, if $\nu \in \sigma_n$, one can obtain a continuation of f to the reflected m-connected circular domain $\Omega_\nu = \rho_\nu\left(\Omega_{\nu_1 \cdots \nu_{n-1}}\right)$ by a finite number of successive reflections. By repeated application of the reflection process, one obtains from the initial function element (f, Ω) a global (many-valued) analytic function $\widehat{f}$ defined on $\mathbb{C}_\infty \backslash cl\{z_{k,\nu}, s_\nu\}$, where $z_{k,\nu}$ and s_ν denote the original prevertices and centers and all of their images under sequences of iterated reflections and cl denotes closure of the set. Any two values $\widehat{f}_r(z)$ and $\widehat{f}_s(z)$ of $\widehat{f}$ at a point $z \in \mathbb{C} \setminus cl\{z_{k,\nu}, s_\nu\}$ are related by the composition of an even number of reflections in lines, and hence $\widehat{f}_s(z) = \alpha \widehat{f}_r(z) + \beta$ for some $\alpha, \beta \in \mathbb{C}$. The preSchwarzian $f''(z)/f'(z)$ is invariant under affine maps $w \longmapsto aw + b$, i.e., $(af(z)+b)''/(af(z)+b)' = f''(z)/f'(z)$. Thus, if one begins with the preSchwarzian of the mapping in Ω, the reflection process yielding the many-valued $\widehat{f}$ also defines a global analytic preSchwarzian, $\widehat{f}''(z)/\widehat{f}'(z)$, which is defined and *single-valued* on $\mathbb{C}\backslash cl\{z_{k,\nu}, s_\nu\}$.

As we shall see, the preSchwarzian is determined by its behaviour on the *singularity set* $\{z_{k,\nu}, s_\nu\}$. Since f extends analytically by reflection across each boundary circle except at the prevertices, it maps a circular arc containing a prevertex $z_{k,i}$ onto a pair of line segments meeting at the vertex $w_{k,i}$ with angle $\alpha_{k,i}\pi$. Hence the function $(f(z) - w_{k,i})^{1/\alpha_{k,i}}$ maps the circular arc onto a straight segment, so by the reflection principle is analytic in a neighborhood of $z_{k,i}$ with a local expansion

$$(f(z) - f(z_{k,i}))^{1/\alpha_{k,i}} = (z - z_{k,i})\, h_{k,i}(z), \tag{3.3}$$

where $h_{k,i}(z)$ is analytic and nonvanishing near z_k. This gives the local expansion

$$f''(z)/f'(z) = \beta_{k,i}/(z - z_{k,i}) + H_{k,i}(z), \quad \beta_{k,i} = \alpha_{k,i} - 1,$$

where $H_{k,i}(z)$ is analytic in a neighborhood of $z_{k,i}$, and $\beta_{k,i}\pi$ is the jump in the tangent angle at the vertex $w_{k,i}$. Recall that elementary geometry of the turning tangent shows that $-1 < \beta_{k,i} \leq 1$ and $\sum_{k=1}^{m} \beta_{k,i} = 2$.

The point at ∞ and its iterated reflections are singularities of the preSchwarzian. This is a departure from the behavior of S–C maps of the disk and annulus to *bounded* polygonal domains. This feature is common to exterior maps and introduces double poles $1/(z-p)^2$ in the integral formula for the S–C mapping formula, [16], p. 329, Thm. 9.9. Since $f(\infty) = \infty$, there is a simple pole of $f(z)$ at infinity and an expansion

$$f(z) = az + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots, \quad a \neq 0$$

at ∞. This gives

$$\frac{f''(z)}{f'(z)} = \frac{\frac{2a_1}{z^3} + \cdots}{a - \frac{2a_1}{z^2} + \cdots} = \frac{2a_1}{az^3} + \left[\frac{1}{z^4}\right]$$

at ∞. Here and elsewhere in the context of a series expansion, we use the standard notation $[(z-\zeta)^k]$ to denote a series in powers of $(z-\zeta)$ that has a factor of $(z-\zeta)^k$.

If $\tilde{f}(z)$ is an analytic continuation by reflection of f across an arc of a circle with center c and radius r, then ∞ reflects to c and

$$\tilde{f}(z) = A\overline{f(\frac{r^2}{\overline{z}-\overline{c}} + c)} + B = A\left(\frac{\overline{a}r^2}{z-c} + \overline{ac} + \overline{a}_0 + \overline{a}_1\frac{z-c}{r^2} + \cdots\right) + B$$

for z near c. This gives

$$\frac{\tilde{f}''(z)}{\tilde{f}'(z)} = \frac{A\left(\frac{2\overline{a}r^2}{(z-c)^3} + \cdots\right)}{A\left(-\frac{\overline{a}r^2}{(z-c)^2} + \cdots\right)} = -\frac{2}{z-c} + b_0 + [(z-c)].$$

A similar calculation shows that the extension of f by reflection through interior circles leads to the same behavior near the reflections s_ν of the centers of the circles C_j,

$$\tilde{f}(z) = \frac{\tilde{A}}{z-s_\nu} + \tilde{B} + [(z-s_\nu)]\,, \quad \frac{\tilde{f}''(z)}{\tilde{f}'(z)} = -\frac{2}{z-s_\nu} + \tilde{b} + [(z-s_\nu)]\,.$$

This can be seen by considering reflections through a circle of radius r and center c, replacing $\rho_c(z) - \rho_c(s_\nu)$ by its conjugate in the expansion of f near $\rho_c(s_\nu)$ and using

$$\frac{r^2}{z-c} - \frac{r^2}{s_\nu - c} = \frac{r^2(z-s_\nu)}{(c-s_\nu)(z-c)} = [(z-s_\nu)]$$

for z near s_ν. This gives the desired expansion around the simple pole $\rho_c(s_\nu)$.

Definition 2. The singularity function $S(z)$ of the global preSchwarzian is the infinite sum of the singular parts of the local expansions of $\widehat{f}''(z)/\widehat{f}'(z)$ at all of its singularities, $z_{k,\nu}$ and s_ν.

Thus one should think that

$$S(z) = \sum_{j=0}^{\infty}\sum_{i=1}^{m}\sum_{\nu\in\sigma_j(i)}\left(\sum_{k=1}^{K_i}\frac{\beta_{k,i}}{z-z_{k,\nu i}} - \frac{2}{z-s_{\nu i}}\right)$$

and with more care formulate the

Definition 3. We have

$$S(z) = \lim_{N\to\infty} S_N(z)$$

where

$$S_N(z) = \sum_{j=0}^{N}\sum_{i=1}^{m}\sum_{\nu\in\sigma_j(i)}\sum_{k=1}^{K_i}\frac{\beta_{k,i}(z_{k,\nu i}-s_{\nu i})}{(z-z_{k,\nu i})(z-s_{\nu i})}$$
$$=\sum_{i=1}^{m}\sum_{k=1}^{K_i}\beta_{k,i}\sum_{\substack{j=0\\ \nu\in\sigma_j(i)}}^{N}\frac{(z_{k,\nu i}-s_{\nu i})}{(z-z_{k,\nu i})(z-s_{\nu i})}.$$

Note the equivalent forms of the summand

$$\sum_{k=1}^{K_i}\frac{\beta_{k,i}(z_{k,\nu i}-s_{\nu i})}{(z-z_{k,\nu i})(z-s_{\nu i})}=\sum_{k=1}^{K_i}\left(\frac{\beta_{k,i}}{z-z_{k,\nu i}}-\frac{\beta_{k,i}}{z-s_{\nu i}}\right)=\left(\sum_{k=1}^{K_i}\frac{\beta_{k,i}}{z-z_{k,\nu i}}\right)-\frac{2}{z-s_{\nu i}}$$

since the $\beta_{k,i}$ sum to 2. We defer to a later Section the convergence proof of the sequence $\{S_N(z)\}$.

The principal idea in our work is that the singularity structure and the following boundary behavior of the preSchwarzian which is shared by $S(z)$ (see Section 5 below) enable one to deduce that $f''(z)/f'(z) = S(z)$ and hence its complete characterization.

Lemma 4. $\mathrm{Re}\{(z-c_j)f''(z)/f'(z)\}_{|z-c_j|=r_j} = -1,\ j = 1,\ldots,m.$

Proof. The tangent angle $\psi(t) = \arg\{ir_je^{it}f'(c_j+r_je^{it})\}$ of the boundary C_j is constant on each of the arcs between prevertices. Hence $\psi'(t) = \mathrm{Re}\{(z-c_j)f''(z)/f'(z)+1\} = 0, |z-c_i| = r_i, z \neq z_{k,j}$. □

We now present our main result.

Theorem 1. *Let $\mathbb{P}$ be an unbounded m-connected polygonal region, $\infty\in\mathbb{P}$ and Ω a conformally equivalent circular domain. Further, suppose $\mathbb{P}$ satisfies the separation property $\Delta < (m-1)^{-1/4}$ for $m > 1$. Then Ω is mapped conformally onto $\mathbb{P}$ by a function of the form $Af(z)+B$, where*

$$f(z) = \int^{z}\prod_{i=1}^{m}\prod_{k=1}^{K_i}\left[\prod_{\substack{j=0\\ \nu\in\sigma_j(i)}}^{\infty}\left(\frac{\zeta-z_{k,\nu i}}{\zeta-s_{\nu i}}\right)\right]^{\beta_{k,i}}d\zeta. \tag{3.4}$$

Here, the turning parameters satisfy $-1<\beta_{k,i}\leq 1$ and $\sum_{k=1}^{m}\beta_{k,i} = 2$. The separation parameter Δ is given (2.4) explicitly in terms of the radii and centers of the circular boundary components of Ω.

Proof. The central idea is to prove that $f''(z)/f'(z) = S(z)$ by means of the argument principle. We use the following two results, whose proofs are postponed to the following two sections of the paper in order to keep the essence of the present proof from being obscured by details of the calculation.

(i) Convergence: $S(z) = \lim_{N\to\infty} S_N(z)$ uniformly on closed subsets of $cl(\Omega)\setminus\{PV\}$, where $\{PV\}$ denotes the set of prevertices on the m circular boundary components of Ω and cl denotes closure of a set.

(ii) B.C.: $\operatorname{Re}\{(z-s_j)S(z)\}_{z\in C_j} = -1,\ j=1,...,m.$

For $z \in (cl\Omega)\setminus\{PV\}$, we define the functions

$$H(z) := \int^z S(\zeta)d\zeta, \quad H_N(z) := \int^z S_N(\zeta)d\zeta, \quad P(z) := e^{H(z)}.$$

We first note that

$$H_N(z) = \int^z S_N(\zeta)d\zeta = \sum_{j=0}^{N}\sum_{i=1}^{m}\sum_{\nu\in\sigma_j(i)}\sum_{k=1}^{K_i}\beta_{k,i}\int^z \frac{1}{\zeta - z_{k,\nu i}} - \frac{1}{\zeta - s_{\nu i}}d\zeta$$

is defined and analytic in Ω since its periods are zero. Indeed, $\int_{C_\tau+} S_N(z)\,dz = 0,\ \tau = 1,...,m$, where $C_\tau+$ is a circle concentric with the boundary circle C_τ with radius slightly larger than that of C_τ, since the residues add out in pairs, and for the "point at infinity" $\lim_{N\to\infty} z^2 S_N(z) = 0$, which eliminates any period over large circles enclosing all of the boundary components of Ω. Furthermore, $H(z)$ is analytic in Ω, since

$$H(z) = \lim_{N\to\infty} H_N(z) = \lim_{N\to\infty}\int^z S_N(\zeta)d\zeta = \int^z S(\zeta)d\zeta, \quad z\in cl(\Omega)\setminus\{PV\}$$

with $S_N(z)\to S(z)$ uniformly on closed subsets of $cl(\Omega)\setminus\{PV\}$.

The next step is to develop a formula for the antiderivative (up to an additive constant)

$$\begin{aligned}
H_N(z) &= \int^z S_N(\zeta)d\zeta = \sum_{j=0}^{N}\int^z \sum_{i=1}^{m}\sum_{\nu\in\sigma_j(i)}\sum_{k=1}^{K_i}\beta_{k,i}\left(\frac{1}{\zeta - z_{k,\nu i}} - \frac{1}{\zeta - s_{\nu i}}\right)d\zeta \\
&= \sum_{j=0}^{N}\sum_{i=1}^{m}\sum_{\nu\in\sigma_j(i)}\sum_{k=1}^{K_i}\beta_{k,i}\int^z \frac{1}{\zeta - z_{k,\nu i}} - \frac{1}{\zeta - s_{\nu i}}d\zeta \\
&= \sum_{j=0}^{N}\sum_{i=1}^{m}\sum_{\nu\in\sigma_j(i)}\sum_{k=1}^{K_i}\beta_{k,i}\log\left(\frac{z - z_{k,\nu i}}{z - s_{\nu i}}\right) \\
&= \sum_{i=1}^{m}\sum_{k=1}^{K_i}\beta_{k,i}\sum_{\substack{j=0\\ \nu\in\sigma_j(i)}}^{N}\log\left(\frac{z - z_{k,\nu i}}{z - s_{\nu i}}\right),
\end{aligned}$$

where each logarithm is the branch that vanishes at $z = \infty$, i.e., $\log 1 = 0$. From the preceding formula, one has

$$P(z) = \lim_{N\to\infty} \exp\{H_N(z)\} = \lim_{N\to\infty} \prod_{i=1}^{m}\prod_{k=1}^{K_i}\left[\prod_{\substack{j=0\\ \nu\in\sigma_j(i)}}^{N}\left(\frac{z-z_{k,\nu i}}{z-s_{\nu i}}\right)\right]^{\beta_{k,i}}$$

and hence the product formula for $P(z)$,

$$P(z) = e^{H(z)} = \prod_{i=1}^{m}\prod_{k=1}^{K_i}\left[\prod_{\substack{j=0\\ \nu\in\sigma_j(i)}}^{\infty}\left(\frac{z-z_{k,\nu i}}{z-s_{\nu i}}\right)\right]^{\beta_{k,i}}. \tag{3.5}$$

Our theorem, $f(z) = A\int^z P(\zeta)\,d\zeta + B$, is equivalent to showing that the quotient

$$Q(z) := \frac{f'(z)}{P(z)} \equiv \text{constant}.$$

To accomplish this, we apply the argument principle to $Q(z)$. First, observe that $P'(z) = H'(z)e^{H(z)} = S(z)P(z)$, that is, $P'(z)/P(z) = S(z)$, and

$$Q' = \frac{f'}{P}\left(\frac{f''}{f'} - \frac{P'}{P}\right) = Q\left(\frac{f''}{f'} - S\right).$$

Then, for $z = c_j + r_j e^{i\theta} \in C_j$, the boundary conditions of Lemma 4 and Theorem 5 on f''/f' and S, respectively, give

$$\begin{aligned}\frac{\partial}{\partial\theta}\arg Q(z) &= \frac{\partial}{\partial\theta}\operatorname{Im}\{\log Q(z)\} = \operatorname{Re}\Big\{(z-c_j)\frac{Q'(z)}{Q(z)}\Big\}\\ &= \operatorname{Re}\Big\{(z-c_j)\left(\frac{f''(z)}{f'(z)} - S(z)\right)\Big\} = 0.\end{aligned}$$

By our construction of $S(z)$, $f''(z)/f'(z) - S(z)$ is continuous on all of C_j, including at the prevertices. Therefore, $\arg Q$ is constant on each of the m boundary circles C_j. Equivalently $Q(C_j)$, the image of C_j, lies on a half-ray emanating from the origin. It is clear by the local behavior (3.3) and formula (3.5) that $Q = f'/P$ is continuous on each C_j and not equal to 0 or ∞ there, since $f', P \neq 0$. Thus for any $w_0 \in \mathbb{C}\backslash Q(C_j)$, $j = 1,\ldots,m$, the winding number of $Q(C_j)$ around w_0 satisfies $n(Q(C_j), w_0) = 0$ for all j. Let C_R be a large circle of radius R centered at the origin containing w_0 and all the C_j's in its interior, and write $C = C_1 \cup\cdots\cup C_m \cup C_R$ with the curves oriented so that the region interior to C_R and exterior to the C_j's is on the left. Since Q has no poles in the region, by the argument principle (for bounded regions), the number of times $Q(z)$ assumes the value w_0 is

$$n(Q(C), w_0) = n(Q(C_1), w_0) + \cdots + n(Q(C_m), w_0) + n(Q(C_R), w_0) = n(Q(C_R), w_0).$$

We now show that $n(Q(C_R, w_0) = 0$. First,

$$n(Q(C_R), w_0) = \frac{1}{2\pi i} \int\limits_{|z|=R} \frac{Q'(z)}{Q(z) - w_0} dz = \frac{1}{2\pi i} \int\limits_{|z|=R} \frac{Q'(z)/Q(z)}{1 - w_0/Q(z)} dz.$$

Recall that $Q'(z)/Q(z) = f''(z)/f'(z) - S(z) = [1/z^3] - [1/z^2] = O(1/z^2)$ for z near ∞, and that $Q(\infty) = f'(\infty)/P(\infty)$ is a finite constant. It suffices to assume $w_0 \neq Q(\infty)$. Then $w_0 \neq Q(z)$ for R sufficiently large, and there are constants $A, B > 0$ such that

$$\left| \int\limits_{|z|=R} \frac{Q'(z)/Q(z)}{1 - w_0/Q(z)} dz \right| \leq A \int\limits_{|z|=R} |Q'(z)/Q(z)||dz| \leq B \int\limits_0^{2\pi} \frac{1}{R^2} R d\theta \to 0$$

as $R \to \infty$. Therefore $n(Q(C), w_0) = 0$, and $Q(z) \neq w_0$ for $w_0 \notin Q(C_j)$ and $w_0 \neq Q(\infty)$. Thus, Q assumes values only on the radial segments $Q(C_j)$ (or $Q(\infty)$) and hence, by the open mapping property of analytic functions, Q must be constant on Ω. □

In the special case when $m = 2$, there is no restrictive separation hypothesis, since then $\Delta < (m-1)^{-1/4} = 1$ is equivalent to the requirement that the two boundary components be nonoverlapping, an obvious necessity for double connectivity of Ω. Clearly, the doubly connected case is easier because there is no exponential doubling of singularities in the reflection process. We conjecture that the result is true for the general case $m > 2$ when $\Delta < 1$, i.e., when the boundary components are disjoint with no additional separation restriction. Note also the remark following the proof of Theorem 3.

The following corollaries show the relative simplicity of the mapping formula for doubly connected unbounded polygonal regions. The second corollary shows that with appropriate normalization of the circular domain, the theta function in Komatu's result (3.2) appears in the formula, and hence removes some of the mystery about the nature of the infinite products.

Corollary 1. *Let $\mathbb{P}$ be an unbounded doubly connected polygonal region, $\infty \in \mathbb{P}$ and Ω a conformally equivalent circular domain. Then Ω is mapped conformally onto $\mathbb{P}$ by a function of the form $Af + B$, where*

$$(3.6)\quad f(z) = \int^z \prod_{k=1}^{K_1} \left[\prod_{\substack{j=0 \\ \nu \in \sigma_j(1)}}^{\infty} \left(\frac{\zeta - z_{k,\nu 1}}{\zeta - s_{\nu 1}} \right) \right]^{\beta_{k,1}} \prod_{k=1}^{K_2} \left[\prod_{\substack{j=0 \\ \nu \in \sigma_j(2)}}^{\infty} \left(\frac{\zeta - z_{k,\nu 2}}{\zeta - s_{\nu 2}} \right) \right]^{\beta_{k,2}} d\zeta.$$

The multi-indices contain only the integers 1 and 2; hence in the first product, there is only one ν in each $\sigma_{j(1)}$, i.e., as j runs through the integers $1, 2, 3, 4, \ldots$

the corresponding sequence of subscripts $\nu 1$ is $1, 21, 121, 2121, \ldots$, respectively. Similarly, in the second product the subscripts $\nu 2$ run $2, 12, 212, 1212$, etc. With some inspired rearranging and intricate manipulations it is possible to get an interesting formula with theta functions which is equivalent to (3.6) and related to (3.2). There is no loss of generality in letting the two boundary circles of Ω be the unit circle C_1 and a circle C_2 with center $c > 1$ and radius r.

Corollary 2. *Let $\mathbb{P}$ be an unbounded doubly connected polygonal region, $\infty \in \mathbb{P}$ and Ω a conformally equivalent circular domain with boundary components the unit circle C_1 and C_2 a circle of radius r and center $c > 1$. Then Ω is mapped conformally onto $\mathbb{P}$ by a function of the form $Af + B$, where*

$$\begin{aligned} f(z) = \int^{z} \prod_{k=1}^{K_1} \left[\Theta\left(\frac{T(\zeta)}{\mu T(z_{1,k})} \right) \right]^{\beta_{k,1}} \prod_{k=1}^{K_2} \left[\Theta\left(\frac{\mu T(\zeta)}{T(z_{2,k})} \right) \right]^{\beta_{k,2}} \\ \times \left\{ \Theta\left(\frac{-pT(\zeta)}{\mu} \right) \Theta\left(\frac{-T(\zeta)}{\mu p} \right) \right\}^{-2} \frac{1-p^2}{(1-p\zeta)^2} d\zeta, \end{aligned} \tag{3.7}$$

where μ is the conformal modulus (2.2) of Ω (and $\mathbb{P}$),

$$T(\zeta) = \frac{\zeta - p}{1 - p\zeta} \quad \text{and} \quad p = \frac{c}{1 + r\mu}.$$

4 Convergence of $S(z)$

In this section, we prove that

$$S(z) = \lim_{N \to \infty} S_N(z)$$

uniformly on closed subsets of $(cl\Omega) \setminus \{PV\}$ when the regions satisfy the separation condition $\Delta < (m-1)^{-1/4}$.

4.1 Convergence for $m = 2$. To illustrate our technique, we first construct the singularity function by a reflection argument for the case $m = 2$. Here $i = 1, 2$ and $\nu = 121 \cdots$ or $\nu = 212 \cdots$. For the map $w = f(z)$ from the exterior of two disks to the exterior of two polygons, the singularity function is

$$S(z) = \sum_{\nu} \left(\sum_{k=1}^{K_1} \frac{\beta_{k,1}}{z - z_{k,\nu 1}} - \frac{2}{z - s_{\nu 1}} \right) + \sum_{\nu} \left(\sum_{k=1}^{K_2} \frac{\beta_{k,2}}{z - z_{k,\nu 2}} - \frac{2}{z - s_{\nu 2}} \right).$$

Our next task is to establish convergence of this expression.

Theorem 2. *For the exterior 2-disk case, $S_N(z)$ converges to $S(z)$ uniformly on closed subsets of $(cl\Omega)\setminus\{PV\}$ by the following estimate:*

$$|S(z) - S_N(z)| = O(\mu_{12}^N).$$

Proof. Using $\sum_{k=1}^{K_i}\beta_{k,i} = 2$, we can write the partial sums $S_N(z)$ of the series for $S(z)$ in a form suitable for establishing convergence of

$$S(z) = \sum_{j=0}^{\infty} A_j(z),$$

where

$$\begin{aligned}A_j(z) &= \sum_{\nu\in\sigma_j(1)}\left(\sum_{k=1}^{K_1}\frac{\beta_{k,1}}{z - z_{k,\nu 1}} - \frac{2}{z - s_{\nu 1}}\right) + \sum_{\nu\in\sigma_j(2)}\left(\sum_{k=1}^{K_2}\frac{\beta_{k,2}}{z - z_{k,\nu 2}} - \frac{2}{z - s_{\nu 2}}\right)\\ &= \sum_{\nu\in\sigma_j(1)}\sum_{k=1}^{K_1}\frac{\beta_{k,1}(z_{k,\nu 1} - s_{\nu 1})}{(z - z_{k,\nu 1})(z - s_{\nu 1})} + \sum_{\nu\in\sigma_j(2)}\sum_{k=1}^{K_2}\frac{\beta_{k,2}(z_{k,\nu 2} - s_{\nu 2})}{(z - z_{k,\nu 2})(z - s_{\nu 2})}.\end{aligned}$$

Letting G be a closed subset of $(cl\Omega)\setminus\{PV\}$, $z \in G$ and applying Lemma 2, we obtain

$$|A_j(z)| \le C(|z_{k,\nu 1} - s_{\nu 1}| + |z_{k,\nu 2} - s_{\nu 2}|) \le C\mu_{12}^j(r_1 + r_2).$$

This estimate establishes the convergence $S(z) = \lim_{N\to\infty} S_N(z)$ as claimed. □

4.2 Convergence for general m.

We let

$$H = cl(\Omega)\setminus\{z_{k,i} : k = 1,\ldots,m, i = 1,\ldots,K_i\}.$$

For $j = 0, 1, 2, \ldots,$ we write

$$A_j(z) = \sum_{i=1}^{m}\sum_{\nu\in\sigma_j(i)}\left(\sum_{k=1}^{K_i}\frac{\beta_{k,i}}{z - z_{k,\nu i}} - \frac{2}{z - s_{\nu i}}\right) = \sum_{i=1}^{m}\sum_{\nu\in\sigma_j(i)}\sum_{k=1}^{K_i}\frac{\beta_{k,i}(z_{k,\nu i} - s_{\nu i})}{(z - z_{k,\nu i})(z - s_{\nu i})};$$

hence, in brief notation,

$$S_N(z) = \sum_{j=0}^{N} A_j(z), \quad S(z) = \lim_{N\to\infty} S_N(z).$$

If G is a closed subset of H, and

$$\begin{aligned}\delta = \delta_G &= \inf_{z\in G}\{|z - z_{k,\nu}|, |z - s_\nu| : k = 1,\ldots,m, \nu\in\sigma\}\\ &= \inf_{z\in G}\{|z - z_{k,i}| : k = 1,\ldots,m, i = 1,\ldots,K_i\},\end{aligned}$$

then $\delta > 0$ and the second expression for δ holds since the $z_{k,\nu}$'s and the s_ν's lie inside the circles.

We have the following

Theorem 3. *For connectivity $m \geq 2$, $S_N(z)$ converges to $S(z)$ uniformly on closed sets $G \subset H$ by the following estimate:*

$$|S(z) - S_N(z)| = O((\Delta^2\sqrt{m-1})^{N+1}),$$

for regions satisfying the separation condition

$$\Delta < 1/(m-1)^{1/4}.$$

Proof. Note that the number of terms in the $A_j(z)$ sum is $O((m-1)^j)$. This exponential increase in the number of terms is the principal difficulty in establishing convergence. Recall that $r_{\nu i}$ is the radius of circle $C_{\nu i}$. We bound $A_j(z)$ for $z \in H$, using the facts $-1 < \beta_{k,i} \leq 1$, $|z_{k,\nu i} - s_{\nu i}| < 2r_{\nu i}$, $K_{\max} := \max_i K_i$, and the Cauchy–Schwarz inequality, as follows:

$$\begin{aligned}
|A_j(z)| &= \Big| \sum_{\nu\in\sigma_j(i)} \sum_{i=1}^{m} \sum_{k=1}^{K_i} \frac{\beta_{k,i}(z_{k,\nu i} - s_{\nu i})}{(z - z_{k,\nu i})(z - s_{\nu i})} \Big| \\
&\leq \sum_{\nu\in\sigma_j(i)} \sum_{i=1}^{m} \sum_{k=1}^{K_i} \frac{|\beta_{k,i}||z_{k,\nu i} - s_{\nu i}|}{|z - z_{k,\nu i}||z - s_{\nu i}|} \\
&\leq \frac{2}{\delta^2} \sum_{\nu\in\sigma_j(i)} \sum_{i=1}^{m} \sum_{k=1}^{K_i} r_{\nu i} \\
&\leq \frac{2K_{\max}}{\delta^2} \sum_{\nu\in\sigma_j(i)} \sum_{i=1}^{m} r_{\nu i} \\
&\leq \frac{2K_{\max}}{\delta^2} \left(\sum_{\nu\in\sigma_j(i)} \sum_{i=1}^{m} r_{\nu i}^2 \right)^{1/2} \left(\sum_{\nu\in\sigma_j(i)} \sum_{i=1}^{m} 1 \right)^{1/2} \\
&= \frac{2K_{\max}}{\delta^2} \left(\sum_{\nu\in\sigma_j(i)} \sum_{i=1}^{m} r_{\nu i}^2 \right)^{1/2} \sqrt{m}(m-1)^{j/2} \\
&\leq \frac{2K_{\max}}{\delta^2} \Delta^{2j} \left(\sum_{i=1}^{m} r_i^2 \right)^{1/2} \sqrt{m}(m-1)^{j/2} \\
&\leq C\Delta^{2j}(m-1)^{j/2}
\end{aligned}$$

by Lemma 3, where $\delta = \delta_G$. Therefore, the series converges if $\Delta^2\sqrt{m-1} < 1$. $\square$

Remark. For $m = 2$, by Proposition 2, we have $\mu_{12} < \Delta^2$, so the results for Theorem 2 are sharper than those of Theorem 3. This suggests the convergence results can be improved for the general case. Our current convergence estimates based on Δ indicate that convergence is fast if the circles are well-separated and slow if some of them are very close to each other.

5 $S(z)$ satisfies the boundary condition

Here we prove that $S(z)$ satisfies the boundary condition

$$\operatorname{Re}\{(z - s_j)\, S(z)\}_{z\in C_j} = -1$$

as claimed in the proof of the main theorem. We use the formulas

$$\operatorname{Re}\left\{\frac{w}{w-1}\right\} = 1/2 \quad \text{for } |w| = 1 \tag{5.1}$$

and

$$\operatorname{Re}\left\{\frac{w}{w-1} + \frac{w^*}{w^*-1}\right\} = 1, \tag{5.2}$$

where w and $w^* = 1/\overline{w}$ are symmetric points with respect to the unit circle.

5.1 The boundary condition for $m = 2$

Theorem 4. *For the unbounded, doubly connected case*

$$\operatorname{Re}\{(z - c_i)S_N(z)\} = -1 + O(\mu_{12}^N)$$

for $z \in C_i$, i.e., $|z - c_i| = r_i$.

Proof. To illustrate our proof, we write out $S_N(z)$ for $N = 2$:

$$\begin{aligned} S_2(z) = &\sum_{k=1}^{K_1} \frac{\beta_{k,1}}{z - z_{k,1}} - \frac{2}{z - s_1} + \sum_{k=1}^{K_2} \frac{\beta_{k,2}}{z - z_{k,2}} - \frac{2}{z - s_2} + \sum_{k=1}^{K_1} \frac{\beta_{k,1}}{z - z_{k,21}} - \frac{2}{z - s_{21}} \\ &+ \sum_{k=1}^{K_2} \frac{\beta_{k,2}}{z - z_{k,12}} - \frac{2}{z - s_{12}} + \sum_{k=1}^{K_1} \frac{\beta_{k,1}}{z - z_{k,121}} - \frac{2}{z - s_{121}} \\ &+ \sum_{k=1}^{K_2} \frac{\beta_{k,2}}{z - z_{k,212}} - \frac{2}{z - s_{212}}. \end{aligned}$$

Next we rearrange $S_2(z)$ in a form convenient for the reflection calculation on the circle $|z - s_1| = r_1$,

$$\begin{aligned}
S_2(z) &= \sum_{k=1}^{K_1} \frac{\beta_{k,1}}{z - z_{k,1}} - \frac{2}{z - s_1} + \sum_{k=1}^{K_2} \frac{\beta_{k,2}}{z - z_{k,2}} - \frac{2}{z - s_2} + \sum_{k=1}^{K_2} \frac{\beta_{k,2}}{z - z_{k,12}} - \frac{2}{z - s_{12}} \\
&\quad + \sum_{k=1}^{K_1} \frac{\beta_{k,1}}{z - z_{k,21}} - \frac{2}{z - s_{21}} + \sum_{k=1}^{K_1} \frac{\beta_{k,1}}{z - z_{k,121}} - \frac{2}{z - s_{121}} \\
&\quad + \sum_{k=1}^{K_2} \frac{\beta_{k,2}}{z - z_{k,212}} - \frac{2}{z - s_{212}} \\
&= \sum_{k=1}^{K_1} \frac{\beta_{k,1}}{z - z_{k,1}} - \frac{2}{z - s_1} + \sum_{k=1}^{K_2} \frac{\beta_{k,2}}{z - z_{k,2}} + \sum_{k=1}^{K_2} \frac{\beta_{k,2}}{z - z_{k,12}} - \frac{2}{z - s_2} - \frac{2}{z - s_{12}} \\
&\quad + \sum_{k=1}^{K_1} \frac{\beta_{k,1}}{z - z_{k,21}} + \sum_{k=1}^{K_1} \frac{\beta_{k,1}}{z - z_{k,121}} - \frac{2}{z - s_{21}} - \frac{2}{z - s_{121}} \\
&\quad + \sum_{k=1}^{K_2} \frac{\beta_{k,2}}{z - z_{k,212}} - \frac{2}{z - s_{212}},
\end{aligned}$$

so that

$$\begin{aligned}
(z - s_1) S_2(z) &= \sum_{k=1}^{K_1} \frac{\beta_{k,1}(z - s_1)}{z - z_{k,1}} - 2 + \sum_{k=1}^{K_2} \frac{\beta_{k,2}(z - s_1)}{z - z_{k,2}} \\
&\quad + \sum_{k=1}^{K_2} \frac{\beta_{k,2}(z - s_1)}{z - z_{k,12}} - 2 \left(\frac{z - s_1}{z - s_2} + \frac{z - s_1}{z - s_{12}} \right) \\
&\quad + \sum_{k=1}^{K_1} \frac{\beta_{k,1}(z - s_1)}{z - z_{k,21}} + \sum_{k=1}^{K_1} \frac{\beta_{k,1}(z - s_1)}{z - z_{k,121}} - 2 \left(\frac{z - s_1}{z - s_{21}} + \frac{z - s_1}{z - s_{121}} \right) \\
&\quad + (z - s_1) \left(\sum_{k=1}^{K_2} \frac{\beta_{k,2}}{z - z_{k,212}} - \frac{2}{z - s_{212}} \right) \\
&= \sum_{k=1}^{K_1} \frac{\beta_{k,1}(z - s_1)/(z_{k,1} - s_1)}{(z - s_1)/(z_{k,1} - s_1) - 1} - 2 \\
&\quad + \sum_{k=1}^{K_2} \beta_{k,2} \left(\frac{(z - s_1)/(z_{k,2} - s_1)}{(z - s_1)/(z_{k,2} - s_1) - 1} \right. \\
&\qquad \left. + \frac{(z - s_1)/(z_{k,12} - s_1)}{(z - s_1)/(z_{k,12} - s_1) - 1} \right) \\
&\quad - 2 \left(\frac{(z - s_1)/(s_2 - s_1)}{(z - s_1)/(s_2 - s_1) - 1} + \frac{(z - s_1)/(s_{12} - s_1)}{(z - s_1)/(s_{12} - s_1) - 1} \right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{K_1}\beta_{k,1}\left(\frac{(z-s_1)/(z_{k,21}-s_1)}{(z-s_1)/(z_{k,21}-s_1)-1}\right.\\
&\qquad\qquad\left.+\frac{(z-s_1)/(z_{k,121}-s_1)}{(z-s_1)/(z_{k,121}-s_1)-1}\right)\\
&-2\left(\frac{(z-s_1)/(s_{21}-s_1)}{(z-s_1)/(s_{21}-s_1)-1}+\frac{(z-s_1)/(s_{121}-s_1)}{(z-s_1)/(s_{121}-s_1)-1}\right)\\
&+(z-s_1)\sum_{k=1}^{K_2}\frac{\beta_{k,2}(z_{k,212}-s_{212})}{(z-z_{k,212})(z-s_{212})}.
\end{aligned}
$$

In the first term of the expression above, $|(z-s_1)/(z_{k,1}-s_1)|=1$ for $|z-s_1|=r_1$. Also, we have paired each $z_{k,\nu}$ and s_ν with their reflections in C_1. Thus, for instance, with $|z-s_1|=r_1$ we have $z_{k,12}-s_1=r_1^2/(\overline{z}_{k,2}-\overline{s}_1)$, and so $(z-s_1)/(z_{k,2}-s_1)$ is the reflection of $(z-s_1)/(z_{k,12}-s_1)$ in the unit circle. Note also that, with $\delta<|z-z_{k,212}|, |z-s_{212}|$, we have for the last term that $|z_{k,212}-s_{212}|\le 2r_{212}=O(\mu_{12}^2)$, since $r_{212}\le\mu_{12}r_{21}\le\mu_{12}^2r_2$ by Lemma 2. The remaining terms are treated similarly. Therefore, using (5.1) and (5.2), the expression above gives for $|z-s_1|=r_1$

$$
\mathrm{Re}\{(z-s_1)S_2(z)\}=1-2+2-2+2-2+O(\mu_{12}^2)=-1+O(\mu_{12}^2).
$$

For the reflection calculation on the circle $|z-s_2|=r_2$, we use the form

$$
S_2(z)=\sum_{k=1}^{K_2}\frac{\beta_{k,2}}{z-z_{k,2}}-\frac{2}{z-s_2}+\cdots+\sum_{k=1}^{K_1}\frac{\beta_{k,1}}{z-z_{k,121}}-\frac{2}{z-s_{121}}.
$$

It is clear that the general case for $S_N(z)$ follows by arranging terms as above. □

5.2 The boundary condition for general m. The following Theorem shows, for general m, that $S(z)$ satisfies the boundary condition for $f''(z)/f'(z)$.

Theorem 5. *If $\Delta<(m-1)^{-1/4}$ then, for $z\in C_i, z\ne z_{k,i}$,*

$$
\mathrm{Re}\{(z-s_i)S_N(z)\}=-1+O((\Delta^2\sqrt{m-1})^N)
$$

and

$$
\mathrm{Re}\{(z-s_i)S(z)\}=-1.
$$

Proof. First, we illustrate the idea of the proof for connectivity $m=3$. The map $w=f(z)$ from the exterior of three disks to the exterior of three polygons has the singularity function

$$S(z) = \sum_{\nu} \left(\sum_{k=1}^{K_1} \frac{\beta_{k,1}}{z - z_{k,\nu 1}} - \frac{2}{z - s_{\nu 1}} \right) + \sum_{\nu} \left(\sum_{k=1}^{K_2} \frac{\beta_{k,2}}{z - z_{k,\nu 2}} - \frac{2}{z - s_{\nu 2}} \right)$$
$$+ \sum_{\nu} \left(\sum_{k=1}^{K_3} \frac{\beta_{k,3}}{z - z_{k,\nu 3}} - \frac{2}{z - s_{\nu 3}} \right),$$

where the $s_{\nu i}$'s are the iterated reflections of the centers c_i of the m boundary circles. We now write out $S_1(z)$ to illustrate our derivation:

$$S_1(z) = \sum_{k=1}^{K_1} \frac{\beta_{k,1}}{z - z_{k,1}} - \frac{2}{z - s_1} + \sum_{k=1}^{K_2} \frac{\beta_{k,2}}{z - z_{k,2}} - \frac{2}{z - s_2} + \sum_{k=1}^{K_3} \frac{\beta_{k,3}}{z - z_{k,3}} - \frac{2}{z - s_3}$$
$$+ \sum_{k=1}^{K_2} \frac{\beta_{k,2}}{z - z_{k,12}} - \frac{2}{z - s_{12}} + \sum_{k=1}^{K_3} \frac{\beta_{k,3}}{z - z_{k,13}} - \frac{2}{z - s_{13}}$$
$$+ \sum_{k=1}^{K_1} \frac{\beta_{k,1}}{z - z_{k,21}} - \frac{2}{z - s_{21}} + \sum_{k=1}^{K_3} \frac{\beta_{k,3}}{z - z_{k,23}} - \frac{2}{z - s_{23}}$$
$$+ \sum_{k=1}^{K_1} \frac{\beta_{k,1}}{z - z_{k,31}} - \frac{2}{z - s_{31}} + \sum_{k=1}^{K_2} \frac{\beta_{k,2}}{z - z_{k,32}} - \frac{2}{z - s_{32}}.$$

Arranging the expressions in a manner similar to the two-disk case, we have

$$(z - s_1)S_1(z) = \sum_{k=1}^{K_1} \frac{\beta_{k,1}(z - s_1)/(z_{k,1} - s_1)}{(z - s_1)/(z_{k,1} - s_1) - 1} - 2$$

$$+ \sum_{k=1}^{K_2} \beta_{k,2} \left(\frac{(z - s_1)/(z_{k,2} - s_1)}{(z - s_1)/(z_{k,2} - s_1) - 1} + \frac{(z - s_1)/(z_{k,12} - s_1)}{(z - s_1)/(z_{k,12} - s_1) - 1} \right)$$
$$-2 \left(\frac{(z - s_1)/(s_2 - s_1)}{(z - s_1)/(s_2 - s_1) - 1} + \frac{(z - s_1)/(s_{12} - s_1)}{(z - s_1)/(s_{12} - s_1) - 1} \right)$$
$$+ \sum_{k=1}^{K_3} \beta_{k,3} \left(\frac{(z - s_1)/(z_{k,3} - s_1)}{(z - s_1)/(z_{k,3} - s_1) - 1} + \frac{(z - s_1)/(z_{k,13} - s_1)}{(z - s_1)/(z_{k,13} - s_1) - 1} \right)$$
$$-2 \left(\frac{(z - s_1)/(s_3 - s_1)}{(z - s_1)/(s_3 - s_1) - 1} + \frac{(z - s_1)/(s_{13} - s_1)}{(z - s_1)/(s_{13} - s_1) - 1} \right)$$
$$+(z - s_1) \left(\sum_{k=1}^{K_1} \frac{\beta_{k,1}(z_{k,21} - s_{21})}{(z - z_{k,21})(z - s_{21})} + \sum_{k=1}^{K_3} \frac{\beta_{k,3}(z_{k,23} - s_{23})}{(z - z_{k,23})(z - s_{23})} \right)$$
$$+(z - s_1) \left(\sum_{k=1}^{K_1} \frac{\beta_{k,1}(z_{k,31} - s_{31})}{(z - z_{k,31})(z - s_{31})} + \sum_{k=1}^{K_2} \frac{\beta_{k,2}(z_{k,32} - s_{32})}{(z - z_{k,32})(z - s_{32})} \right)$$

$$= \sum_{k=1}^{K_1} \frac{\beta_{k,1}(z-s_1)/(z_{k,1}-s_1)}{(z-s_1)/(z_{k,1}-s_1)-1} - 2$$

$$+\sum_{k=1}^{K_2} \beta_{k,2}\left(\frac{(z-s_1)/(z_{k,2}-s_1)}{(z-s_1)/(z_{k,2}-s_1)-1} + \frac{(z-s_1)/(\rho_1(z_{k,2})-s_1)}{(z-s_1)/(\rho_1(z_{k,2})-s_1)-1}\right)$$

$$-2\left(\frac{(z-s_1)/(s_2-s_1)}{(z-s_1)/(s_2-s_1)-1} + \frac{(z-s_1)/(\rho_1(s_2)-s_1)}{(z-s_1)/(\rho_1(s_2)-s_1)-1}\right)$$

$$+\sum_{k=1}^{K_3} \beta_{k,3}\left(\frac{(z-s_1)/(z_{k,3}-s_1)}{(z-s_1)/(z_{k,3}-s_1)-1} + \frac{(z-s_1)/(\rho_1(z_{k,3})-s_1)}{(z-s_1)/(\rho_1(z_{k,3})-s_1)-1}\right)$$

$$-2\left(\frac{(z-s_1)/(s_3-s_1)}{(z-s_1)/(s_3-s_1)-1} + \frac{(z-s_1)/(\rho_1(s_3)-s_1)}{(z-s_1)/(\rho_1(s_3)-s_1)-1}\right)$$

$$+(z-s_1)\left(\sum_{k=1}^{K_1} \frac{\beta_{k,1}(z_{k,21}-s_{21})}{(z-z_{k,21})(z-s_{21})} + \sum_{k=1}^{K_3} \frac{\beta_{k,3}(z_{k,23}-s_{23})}{(z-z_{k,23})(z-s_{23})}\right)$$

$$+(z-s_1)\left(\sum_{k=1}^{K_1} \frac{\beta_{k,1}(z_{k,31}-s_{31})}{(z-z_{k,31})(z-s_{31})} + \sum_{k=1}^{K_2} \frac{\beta_{k,2}(z_{k,32}-s_{32})}{(z-z_{k,32})(z-s_{32})}\right).$$

The truncation error will be given by the terms in the last two lines. The real part of the initial terms can be calculated with a reflection argument as in the doubly connected case. For the first term, let $w = (z-s_1)/(z_{k,1}-s_1)$. Note that $|w| = 1$ for $z \in C_1$. Then by (5.1), we have

$$\mathrm{Re}\left\{\frac{(z-s_1)/(z_{k,1}-s_1)}{(z-s_1)/(z_{k,1}-s_1)-1}\right\} = \mathrm{Re}\left\{\frac{w}{w-1}\right\} = \frac{1}{2}.$$

For the second group of terms, we have, for instance, $w = (z-s_1)/(z_{k,2}-s_1)$ and $w^* = (z-s_1)/(\rho_1(z_{k,2})-s_1)$. Using (5.2), this gives

$$\mathrm{Re}\left\{\frac{(z-s_1)/(z_{k,2}-s_1)}{(z-s_1)/(z_{k,2}-s_1)-1} + \frac{(z-s_1)/(\rho_1(z_{k,2})-s_1)}{(z-s_1)/(\rho_1(z_{k,2})-s_1)-1}\right\}$$
$$= \mathrm{Re}\left\{\frac{w}{w-1} + \frac{w^*}{w^*-1}\right\} = 1.$$

To bound the error in the final terms, note that $|z_{k,ij} - s_{ij}| \leq 2r_{ij}$. Then using Lemma 3 for $|z-s_1| = r_1$, we obtain the error bound

$$\frac{2r_1}{\delta_{\{z\}}^2}(r_{21}+r_{23}+r_{31}+r_{32}) \leq C\sqrt{r_{21}^2+r_{23}^2+r_{31}^2+r_{32}^2}\sqrt{4}$$
$$\leq C\sqrt{r_2^2+r_3^2}\sqrt{4}\Delta^2 = O(\sqrt{3-1}(3-1)^{1/2}\Delta^2).$$

Therefore,

$$\begin{aligned}\operatorname{Re}\{(z-s_1)S_1(z)\} &= 1-2+2(2-2)+O(\sqrt{3-1}(3-1)^{1/2}\Delta^2)\\ &= -1+O(\sqrt{m-1}(\Delta^{2N}(m-1)^{N/2})\end{aligned}$$

when $m=3$ and $N=1$. The general case is similar, using Proposition 1 to group terms related by reflection ρ_p through C_p with $z \in C_p$ as follows:

$$(z-s_p)S_N(z) = \sum_{k=1}^{K_p} \frac{\beta_{k,p}(z-s_p)/(z_{k,p}-s_p)}{(z-s_p)/(z_{k,p}-s_p)-1} - 2$$

$$+\sum_{j=0}^{N-1}\sum_{i=1}^{m}\sum_{\substack{\nu\in\sigma_j(i),\\ \nu i,\nu_1\neq p}}\sum_{k=1}^{K_i}\beta_{k,i}\left(\frac{(z-s_p)/(z_{k,\nu i}-s_p)}{(z-s_p)/(z_{k,\nu i}-s_p)-1} + \frac{(z-s_p)/(\rho_p(z_{k,\nu i})-s_p)}{(z-s_p)/(\rho_p(z_{k,\nu i})-s_p)-1}\right)$$

$$-2\sum_{j=0}^{N-1}\sum_{i=1}^{m}\sum_{\substack{\nu\in\sigma_j(i)\\ \nu i,\nu_1\neq p}}\left(\frac{(z-s_p)/(s_{\nu i}-s_p)}{(z-s_p)/(s_{\nu i}-s_p)-1} + \frac{(z-s_p)/(\rho_p(s_{\nu i})-s_p)}{(z-s_p)/(\rho_p(s_{\nu i})-s_p)-1}\right)$$

$$+(z-s_p)\sum_{\substack{j=1,\\ j\neq p}}^{m}\sum_{i=1}^{m}\sum_{j\nu\in\sigma_N(i)}\left(\sum_{k=1}^{K_i}\frac{\beta_{k,i}(z_{k,j\nu i}-s_{j\nu i})}{(z-z_{k,j\nu i})(z-s_{j\nu i})}\right).$$

The first term and the terms grouped by reflection through C_p are handled as before. The final $m-1$ terms, all lying inside circles C_i, $i \neq p$, approximate the truncation error and are estimated by

$$\sum_{\nu\in\sigma_{n+1}} r_\nu^2 \le \Delta^{4N}\sum_{i=1}^{m} r_i^2.$$

This gives our final result

$$\begin{aligned}\operatorname{Re}\{(z-s_p)S_N(z)\} &= 1-2+(m-1)(2-2)+(m-1)^2(2-2)\\ &\quad+\cdots+(m-1)^{N-1}(2-2)+O(\sqrt{m-1}(m-1)^{N/2}\Delta^{2N})\\ &= -1+O(\sqrt{m-1}(\Delta^{2N}(m-1)^{N/2}).\end{aligned}$$

□

6 Examples

In this section, we give some graphical examples of specific mappings computed with our formula. These examples are done with a primitive computational procedure and are intended only to give a preliminary indication of what can be obtained by computing with our mapping formula. In particular, our computatonal work here does not yield the conformal invariants of a given polygonal region. This will be an automatic byproduct of a complete numerical implementation that is in progress. In this work, the polygonal domain is specified and the side lengths of the polygons are used as constraints that force the numerical implementation to produce the conformally equivalent circle domain along with the mapping.

For the present paper, we have developed an initial MATLAB code to demonstrate computationally the feasibility and correctness of our formula for some simple examples. One such example is given (for $m = 3$) of the mapping of the exterior of three circles, Figure 3, to the exterior of two oblique slits and a rectangle (the solid lines) in Figure 2. For this example, we have put the parameters into the product formula and performed two levels of reflections, i.e., truncating the products after $N = 2$. Since the computational circles are well-separated, the sum of the squares of the radii decrease very rapidly, as seen in Table 1 and illustrated in Figure 3. The trapezoidal rule is used for the numerical integration. This gives rough accuracy for the turning angles we have chosen. The map is evaluated on the boundary circles, 3 concentric circles and on Cartesian grid lines (dashed lines in Figure 2). Note that integrating around closed curves in the computational region results in (approximately) closed curves in the image plane, confirming the (near) single-valuedness of our truncated mapping formula. A quick "reality check" of the accuracy of our approximation can be made by looking at a typical factor in the infinite product

$$\frac{\zeta - z_{k,\nu i}}{\zeta - s_{\nu i}} = 1 + \frac{s_{\nu i} - z_{k,\nu i}}{\zeta - s_{\nu i}}$$

and the extremely rapid shrinking of the circles in the first reflection as shown in Figure 3 and realizing that $|s_{\nu i} - z_{k,\nu i}|$ is bounded by the diameter of the νi circle. This supports the belief that the factors for $N > 2$ must be near 1 to very high precision and also explains why the sides of the approximate image polygons appear (to the eye) to be straight rather than somewhat "wavy".

The parameters for the example in Figure 2 are

$$c_1 = 1 + 2i, \quad r_1 = 0.5, \quad \beta_{1,1} = \beta_{2,1} = 1, \quad z_{1,1} = c_1 + r_1 e^{i3\pi/4},$$
$$z_{2,1} = c_1 + r_1 e^{i7\pi/4}, \quad c_2 = 3 + i, \quad r_2 = 0.4, \quad \beta_{1,2} = \beta_{2,2} = \beta_{3,2} = \beta_{4,2} = 0.5,$$
$$z_{1,2} = c_2 + r_2 e^{i3\pi/8}, \quad z_{2,2} = c_2 + r_2 e^{i5\pi/8}, \quad z_{3,2} = c_2 + r_2 e^{i11\pi/8},$$
$$z_{4,2} = c_2 + r_2 e^{i13\pi/8}, \quad c_3 = 3 + 3i, \quad r_3 = 0.2, \quad \beta_{1,3} = \beta_{2,3} = 1,$$
$$z_{1,3} = c_3 + r_3 e^{i\pi/8}, \quad z_{2,3} = c_3 + r_3 e^{i9\pi/8}.$$

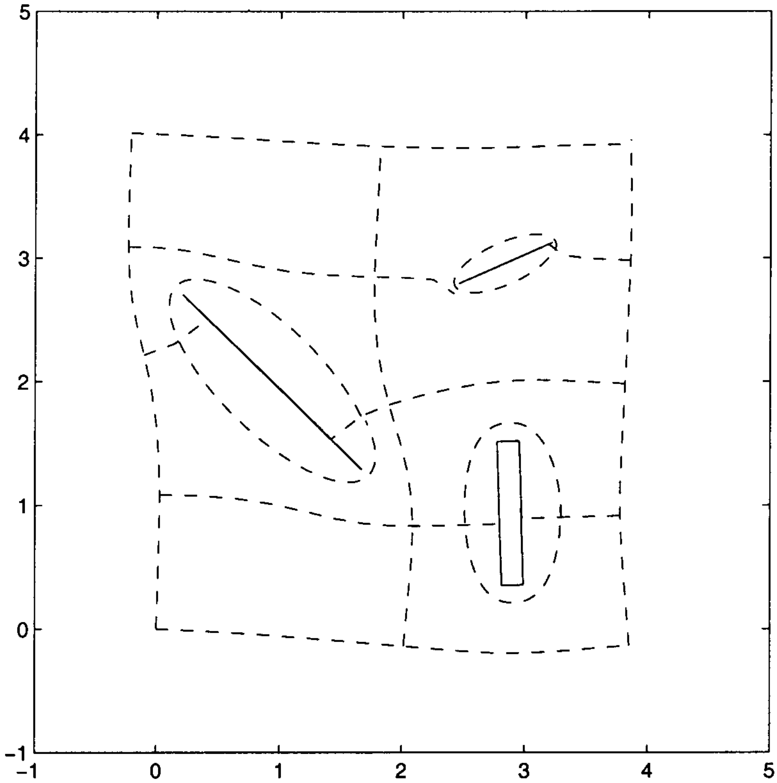

Figure 2. Triply connected S–C image of Figure 3 domain.

Figure 3 illustrates the extremely rapid rate of decrease in the size of the circles due to the (visually) large separation of the three original boundary circles. This rapid change is confirmed by the numerical results in Table 1 showing that the decrease in the areas of the reflected circles is much faster than predicted by our theoretical estimate with Δ.

N	$\Delta^{4N}\sum_{i=1}^{3} r_i^2$	$\sum_{\lvert\nu\rvert=N+1} r_\nu^2$
0	.45	.45
1	$.12 \cdot 10^{-1}$	$.91 \cdot 10^{-3}$
2	$.31 \cdot 10^{-3}$	$.19 \cdot 10^{-5}$

Table 1. Rapid decrease of areas of reflected circles.

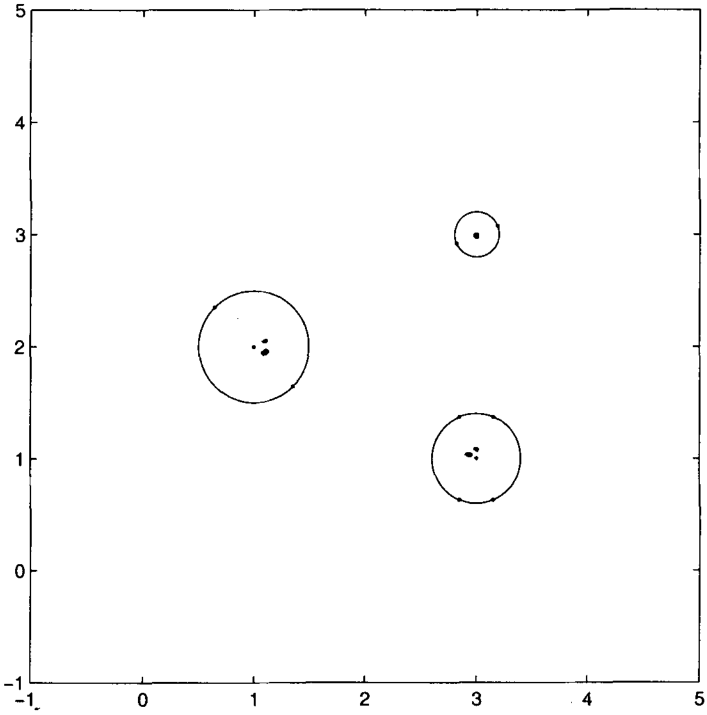

Figure 3. Reflected circles in the exterior of the preimage of Figure 2.

Remark. Since our main theorem shows that $f''/f' = S$, we have $S(z) = [1/z^3]$ at ∞. As a consequence, the coefficient of $1/z^2$ in the expansion at ∞ of $S(z)$ must be zero. Thus, from the series expansion in Definition 3, one has the necessary *side condition* on the prevertex and turning parameters

$$\sum_{i=1}^{m}\sum_{k=1}^{K_i}\sum_{\substack{j=0\\ \nu\in\sigma_j(i)}}^{\infty} \beta_{k,i}\left(z_{k,\nu i}-s_{\nu i}\right) = \lim_{N\to\infty}\sum_{i=1}^{m}\sum_{k=1}^{K_i}\sum_{\substack{j=0\\ \nu\in\sigma_j(i)}}^{N} \beta_{k,i}\left(z_{k,\nu i}-s_{\nu i}\right) = 0.$$

This side condition is necessary to complete the system of equations for the numerical solution of the parameter problem. The simpler, well-known side condition $\sum_1^K \beta_k z_k = 0$ appears in the S–C mapping of the exterior of the unit disk to the exterior of a single polygon for the same reason. In our numerical example, the side condition has not been imposed but is satisfied with an error of about 0.01. This is roughly the level at which plots of the images of the circles fail to close, indicating a "small lack of single-valuedness".

Remarks. Conformal mappings between general multiply connected unbounded regions can be obtained by combining the methods of this paper with those of [4]. We mentioned earlier that parallel slit domains are a natural type of polygonal domain to use in conjunction with our results. For example, if one has a parallel slit domain (in the w-plane) with slits parallel to the real axis, then a uniform potential flow parallel to the real axis can be transplanted back to a conformally equivalent circle domain with our S–C mapping $w = f(z)$. The streamlines of the corresponding flow around the preimage circles in the z-plane are the curves $\operatorname{Im}\{f(z)\} = \text{constant}$. Combining this with the numerical methods of [4], one then has the streamlines of uniform flow around a finite set of rather general obstacles in the plane. This is just one primitive example of many applications of the S–C conformal map of multiply connected domains. Other areas of application include problems in electrostatic fields, cf. [1], and the complex analysis problem of (numerically) determining the conformal invariants of a multiply connected polygonal domain (since one can read the invariants of a circular domain from the center and radius values).

Acknowledgement: The work of the first two authors was partially supported by NSF EPSCoR grant No. 9874732.

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(Received January 26, 2003)