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Extremal distance, harmonic measure and numerical conformal mapping

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Abstract

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Estimates of extremal distance and harmonic measure are used to show how the geometric properties of a simply connected domain influence the boundary distortion of a conformal map from the unit disk to the domain. Numerical examples and remarks on the conditioning of numerical conformal mapping methods are included. A sharp estimate is given of the exponential ill-conditioning, known as the crowding phenomenon, which occurs for slender regions.

Keywords: Numerical conformal mapping; crowding; extremal length; harmonic measure.

1. Introduction

Many of the most important methods for the numerical computation of a conformal map proceed by constructing the boundary correspondence function. Thus extreme stretching or compressing of boundary sets by a conformal mapping can increase the discretization error in numerical conformal mapping. In this paper we consider a conformal map f from the open unit disk D to the simply connected domain Ω bounded by a piecewise smooth, closed curve $\partial\Omega$. We wish to study how the boundary distortion by f is influenced by elementary geometric characteristics of the domain and by the location of $f(0)$. There are two ways to study these distortions. One is to look for upper and lower bounds for $|f'|$ as is done in [8,25,26]. The other is to apply certain estimates of harmonic measure of boundary sets in terms of extremal

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distance. By studying how boundary sets may be compressed and expanded under conformal maps, we show how the slenderness of Ω can influence the conditioning of the problem and, hence, the "resolution" or discretization error in numerical approximations to conformal maps. Roughly, there are two ways to ill-condition the problem: (i) by "pinching" or "slitting" the domain, which creates "algebraically bad" distortions, as illustrated in estimate (5) and Examples 5.1, 5.3 and 5.4, and (ii) by "stretching" the domain, which creates "exponentially bad" distortions, as illustrated in estimates (2), (3), (4b) and Examples 5.2 and 5.5. For highly elongated domains with aspect ratios on the order of 10 to 1 or more, this latter exponential distortion, known as the *crowding phenomenon*, may make the numerical problem impossible to solve. This phenomenon was first noticed by Gaier [9, p.179]. It was also observed in numerical experiments in [10,16] and has been discussed further in [4,5,8,25,26]. Methods which circumvent the problem to some extent, by choosing more appropriate computational domains, have been proposed in [7,13,18–20].

The paper is organized as follows. In Section 2, we recall the properties of harmonic measure and extremal distance and their relation to conformal mapping. In Section 3, we apply an inequality by Pfluger to obtain a sharp estimate of the exponential crowding for regions with elongated sections. In Section 4, we consider two estimates, one due to Dubiner, that illustrate the less severe algebraic distortions. Finally, in Section 5, we give several explicit and numerical examples and make some remarks on numerical methods.

2. Harmonic measure and extremal distance

We shall review a few of the basic facts about harmonic measure. The reader is referred to [1, Chapters 3 and 4] for more details.

Let Ω be a Jordan domain in the z-plane, and let E be an arc on the boundary $\partial\Omega$.

Definition 2.1. The *harmonic measure of* E with respect to Ω, $\omega(z) = \omega(z, E, \Omega)$, is the unique bounded harmonic function on Ω with boundary values 1 at interior points of E and 0 at interior points of $\partial\Omega - E$.

If $z = f(w)$ maps D conformally onto the Jordan domain Ω and $g = f^{-1}$, then

$$\omega(w, g(E), D) = \omega(f(w), E, \Omega),$$

since harmonic functions are preserved by conformal maps. The boundary arc E maps to an arc $g(E)$ on the unit circle. It is a familiar and easy consequence of the mean value property of harmonic functions that

$$\operatorname{meas}(g(E)) = 2\pi\omega(0, g(E), D) = 2\pi\omega(f(0), E, \Omega), \tag{1}$$

where meas denotes linear measure on the unit circle.

Thus, estimates of the harmonic measure $\omega(z_0, E, \Omega)$, $z_0 = f(0)$, can be used to assess the distortion of the boundary set E under conformal mapping by f. Extremal lengths of curve families and extremal distances between sets are conformally invariant notions and hence one can often find rather precise connections with the conformally invariant harmonic measure [2,3]. We will estimate extremal distance in terms of geometric properties of Ω and $\partial\Omega$. These estimates then will yield corresponding bounds on the harmonic measure.

Now we recall the notions of extremal length and distance. Let Γ be a set of locally rectifiable arcs γ in Ω. We consider the family of Riemannian metrics $\mathrm{d}s = \rho\,|\mathrm{d}z|$ which includes the Euclidean metric.

Definition 2.2. The *extremal length* of Γ is

$$\lambda(\Gamma, \Omega) = \sup_{\rho} \frac{\left(\inf_{\gamma\in\Gamma} \int_{\gamma} \rho\,|\mathrm{d}z|\right)^2}{\iint_{\Omega} \rho^2 \,\mathrm{d}x\,\mathrm{d}y},$$

where the supremum is over all nonnegative, Borel measurable $\rho = \rho(x, y)$ such that $0 < \iint_{\Omega} \rho^2 \,\mathrm{d}x\,\mathrm{d}y < \infty$.

Under a conformal map f the metric ρ will transform as $\mathrm{d}s = \rho\,|\mathrm{d}z| = \rho'\,|\mathrm{d}f|$ where $\rho = \rho/|f'|$. Thus, we see that extremal length is a *conformal invariant*, which may be thought of as *length*^{2}*/area*.

Next, let K' and K be two disjoint sets in the closure of Ω. Let Γ be the family of arcs in Ω connecting K' and K. Then we have the following definition.

Definition 2.3. The *extremal distance* between K' and K is just the extremal length of Γ,

$$\lambda(K', K, \Omega) = \lambda(\Gamma, \Omega);$$

see [1, p.52].

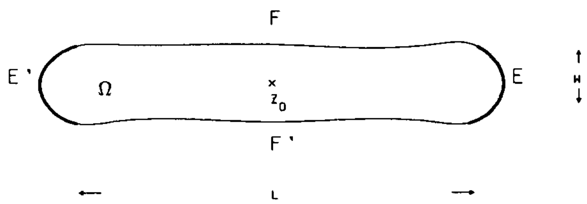

Fig. 1. Quadrilateral.

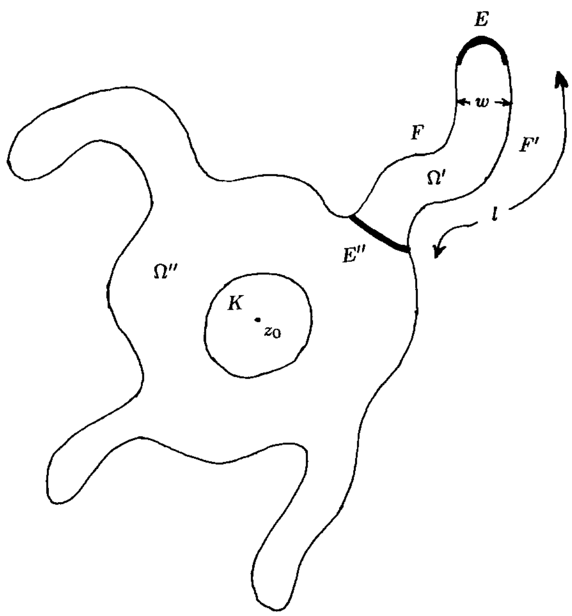

Fig. 2. Region for Theorem 3.1.

For the "quadrilateral" (see [1, p.52]) in Fig. 1 with marked sides E and E', the extremal distance $\lambda(E', E, \Omega)$ is also the *modulus* of this quadrilateral. The region Ω in Fig. 1 with sides E, F, E', F' may be conformally mapped onto a rectangle R with corresponding sides $\tilde{E}$, $\tilde{F}$, $\tilde{E}'$, $\tilde{F}'$, when the moduli are equal, that is, if and only if

$$\lambda(E', E, \Omega) = \lambda(\tilde{E}', \tilde{E}, R) = \frac{a}{b},$$

where $a = \operatorname{dist}(\tilde{E}, \tilde{E}')$ and $b = \operatorname{dist}(\tilde{F}, \tilde{F}')$ are the dimensions of R.

Let A be the area of Ω, $l = d(E', E, \Omega)$ the *interior Euclidean distance* from E' to E in Ω, and $w = d(F', F, \Omega)$. If the quadrilateral Ω is a rectangle, then $\lambda(E', E, \Omega) = l/w$. If z_0 is the center of Ω and $l \gg w$, then, using the properties of elliptic integrals which arise in the Schwarz–Christoffel map from the disk to the rectangle, it is well known [13,18] that

$$\omega(z_0, E, \Omega) \sim \frac{4}{\pi}\exp\left(-\frac{\pi l}{2w}\right). \tag{2}$$

In [4] it was noted that for a "slender" region Ω with E' and E at the "ends" we have $l \gg w$ and $A \approx lw$. Thus Rengel's inequality [1, p.54] or [14, p.22] gives

$$\lambda(E', E, \Omega) \approx \frac{l}{w},$$

and so Ω maps to the rectangle with aspect ratio $\approx l/w$. Then, assuming that a point z_0 near the "center" of Ω is mapped to the center of the rectangle, it was observed that (2) gives

$$\omega(z_0, E, \Omega) \approx \frac{4}{\pi}\exp\left(-\tfrac{1}{2}\pi\lambda(E', E, \Omega)\right), \tag{3}$$

for $\lambda(E', E, \Omega) = l/w \gg 1$. This expresses the severe compression or "crowding" of the image of E (and E') on the unit circle for slender regions of large aspect ratio; see Example 5.2 for an explicit map illustrating this situation.

Ideally, for our applications, we would like to replace $\lambda(E', E, \Omega)$ by "$\lambda(z_0, E, \Omega)$", where $f(0) = z_0$. However, as noted in [1, p.78], if one of the sets E' shrinks to a point z', then $\lambda(E', E, \Omega)$ tends to ∞. We shall see that this problem is avoided in an inequality by Pfluger by using a small compact set containing the point.

3. A sharp estimate of the exponential crowding

Here we use a theorem due to Pfluger to give an estimate of the exponential crowding for regions with elongated sections. The estimate applies generally to any boundary set E at the end of a "finger" Ω' of length l and width w protruding from Ω, as in Fig. 2, and also yields the known exponential behavior for the elongated regions in our examples.

Theorem 3.1. *Let the domain $\Omega = \Omega' \cup E'' \cup \Omega''$ where $\Omega' \cap \Omega'' = \emptyset$ and Ω' is a quadrilateral with sides E, E'', F, F' with $E \cup F \cup F' \subset \partial\Omega$, a piecewise smooth curve; see Fig. 2. Fix $z_0 \in \Omega''$ and let $l = d(E'', E, \Omega')$ and $w = d(F', F, \Omega')$. Let A' be the Euclidean area of Ω'. Then*

$$\omega(z_0, E, \Omega) \leqslant C \exp\left(-\frac{\pi l^2}{A'}\right), \tag{4a}$$

and, if $A' = lw(1+\epsilon)$,

$$\omega(z_0, E, \Omega) \leqslant C \exp\left(-\frac{\pi l}{w(1+\epsilon)}\right), \tag{4b}$$

where the constant C *depends only on* z_0.

Proof. Fix $z_0 \in \Omega''$ and a number $\delta \in (0, 1)$ and let $K = \{z\colon\ |z - z_0| \leqslant \delta \operatorname{dist}(z_0, \partial\Omega'')\}$. Then with [21, Theorem 1] (Pfluger), one can obtain the following inequality [2, Eq. 6.3, p.304]:

$$\omega(z_0, E, \Omega) \leqslant C\, \mathrm{e}^{-\pi\lambda(K,E,\Omega)},$$

where the constant C depends only on z_0 and K for a fixed Ω, Ω', Ω'', and does not depend on the boundary set E. With Definition 2.2 and the metric ρ that is 0 in Ω'' and 1 in Ω', we obtain the inequality

$$\lambda(K, E, \Omega) \geqslant \frac{l^2}{A'}.$$

Combining this with Pfluger's inequality yields the results. □

From (4a), (4b) we make the following observations.

(i) Suppose Ω is a slender quadrilateral with sides E, E', F, F' and $l = d(E', E, \Omega)$, $w = d(F', F, \Omega)$. We fix z_0 near the center of Ω, $d(z_0, E) \approx \frac{1}{2}l$, and assume that there is a crosscut E'' separating z_0 and E with $d(E'', E, \Omega) = \frac{1}{2}(1-\epsilon)l$, $A' = \frac{1}{2}(1+\epsilon')lw$ for small positive ϵ and small ϵ'. Then with Pfluger's theorem and the reasoning of Theorem 3.1, we obtain

$$\omega(z_0, E, \Omega) \leqslant C \exp\left(-\frac{\pi l}{2w}\left(\frac{1-\epsilon}{1+\epsilon'}\right)\right) \leqslant C_1 \exp\left(-\frac{\pi l}{2w}\right).$$

The map of the unit disk onto a rectangle shows that the exponent is sharp, cf. (2) above. Wegmann [25] also gives estimates with the correct exponential behavior. His approach is different and somewhat complementary to ours in that he gives a lower bound for the supremum norm of the derivative of the map from the unit disk to an elongated region by combining the known behavior of certain explicit maps with a generalization of the Schwarz lemma. His results are somewhat less general than ours in that his regions must be contained in a smallest rectangle of length l and width w. However, he does give a more detailed discussion of the constant C for certain regions. See Examples 5.2 and 5.5.

(ii) If a portion of Ω between z_0 and E is a channel whose width $\theta = \theta(x)$ is parametrized by x, for instance, with $a \leqslant x \leqslant b$, then the *Ahlfors Distortion Theorem* [1, pp. 56, 57], [2, Sections 1 and 4] may be used to give a more precise estimate

$$\lambda(K, E, \Omega) \geqslant \int_a^b \frac{\mathrm{d}x}{\theta(x)}.$$

(iii) For a slender region (4b) may be interpreted as a St. Venant-like principle, expressing the exponential decay-of-influence of boundary data at E on the value of a harmonic function at z_0 as a function of the distance from z_0 to E; see, e.g., [11].

(iv) Dubiner gives similar estimates in his thesis [8]. His estimates appear to apply to more general domains, but they are not as accessible as ours and do not appeal to Pfluger's theorem. Our study of Dubiner's work has provided a strong motivation for the present work.

(v) In [3], Beurling gives two definitions of conformally invariant extremal distance between a point and a boundary set. He also gives estimates of harmonic measure in terms of his notions of extremal distance. We have used Beurling's results to exhibit the exponential crowding; however, we were not able to derive sharp estimates with his results.

(vi) We note here that there is an extensive literature on the boundary behavior of conformal maps with relations to topics such as probability theory. An introduction to that literature is given in [15] and the references contained therein. The results there deal with distortions of boundary sets and boundaries that are much more pathological than typically occur in computational practice, such as fractal boundaries.

4. A theorem of Dubiner and another estimate

Though the crowding may be severe for slender regions as Example 5.2 and estimates (2), (3) and (4b) demonstrate, the distortions are less severe for regions which are not elongated but pinched as in Example 5.1, 5.3 and 5.4. These explicit and computational examples are given in Section 5 along with bounds on the derivatives or the distortions of certain boundary arcs. The boundary curves are all scaled, so the maximum distance from $f(0)$ to $\partial\Omega$ is O(1). The minimum distance from $f(0)$ to $\partial\Omega$—the "thinness"—is denoted by α. The information on the derivatives may be interpreted as follows. Let E be a (small) boundary arc at a distance O(1) from $f(0)$. Then, in Examples 5.1, 5.3 and 5.4 and the Cassini oval,

$$\omega(f(0), E, \Omega) = \mathrm{O}(\alpha^k), \quad k > 0.$$

That is, $g(E)$ is compressed like $\mathrm{O}(\alpha^k)$, but not so severely as for a slender region as in Example 5.2. Similarly, let F be a (small) boundary arc a distance $\mathrm{O}(\alpha)$ from $f(0)$. Then in all cases

$$\omega(f(0), F, \Omega) = \mathrm{O}(\alpha^{-k}), \quad k > 0.$$

That is, $g(F)$ is expanded like $\mathrm{O}(\alpha^{-k})$, but not more severely. The value of k may depend not just on the "thinness" but also on "higher order" or local effects such as curvature, corners, etc.

Below we state and illustrate a theorem of Dubiner on the compression of sets E, but first we recall a classical results that indicates the $\mathrm{O}(\alpha^{-1})$ behavior of $g'(0)$ which is seen in our examples.

Schwarz Lemma. *Let f map the unit disk D conformally to a domain Ω contained in D, with $f(0) = 0$ and with boundary $\partial\Omega$ a distance α from the origin. Then*

$$\alpha \leqslant |f'(0)| \leqslant 1,$$

or for $g = f^{-1}$,

$$1 \leqslant |g'(0)| \leqslant \alpha^{-1}.$$

Proof. Since $|f(z)| \leqslant 1$, $f'(0) \neq 0$, and $f(z)/z \neq 0$ is analytic for $|z| < 1$, the maximum and minimum principles imply that $\alpha \leqslant |f(z)/z| \leqslant 1$. □

Dubiner [8, Theorem 8.1] says that if the maximum distance from $f(0)$ to $\partial\Omega$ is O(1) and the minimum distance from $f(0)$ to $\partial\Omega$ is α, then for boundary arcs E a distance O(1) from $f(0)$ we have, roughly,

$$\omega(f(0), E, \Omega) \leqslant 2\sqrt{\alpha}\,. \tag{5}$$

We will illustrate this in Example 5.4.

5. Remarks on numerical methods and examples

The effects of the geometry of Ω on Fourier series methods is discussed, for instance, in [5] and, for exterior regions, in [6]. A "rule of thumb" due to Zemach [26] says that for even an order-of-magnitude approximation to f one needs to take at least $N = \mathrm{O}(\max |f'|)$ Fourier coefficients. To see this, suppose $\partial\Omega$ is a smooth curve parametrized by arclength σ with total arclength L. Let $\Delta t = 2\pi/N$. Find the smallest N such that $\Delta\sigma/\Delta t \approx \mathrm{d}\sigma/\mathrm{d}t$ for all t. Then $|f'(\mathrm{e}^{it})| = \mathrm{d}\sigma/\mathrm{d}t \approx \Delta\sigma/\Delta t \leqslant NL/2\pi$ for all t. Therefore, for N large enough so that f'_N, the derivative of the truncated series, is a good approximation to f', we require $N \geqslant 2\pi(\max |f'|)/L$. For certain explicit cases this rule can be stated more precisely, as shown in [5]. The severe stretching in maps to slender regions limits the usefulness of Fourier series methods which attempt to approximate the Taylor series of f. These observations are illustrated in Examples 5.1 and 5.2. The corresponding Figs. 3 and 4, respectively, show the maps to the regions approximated with Wegmann's method [24].

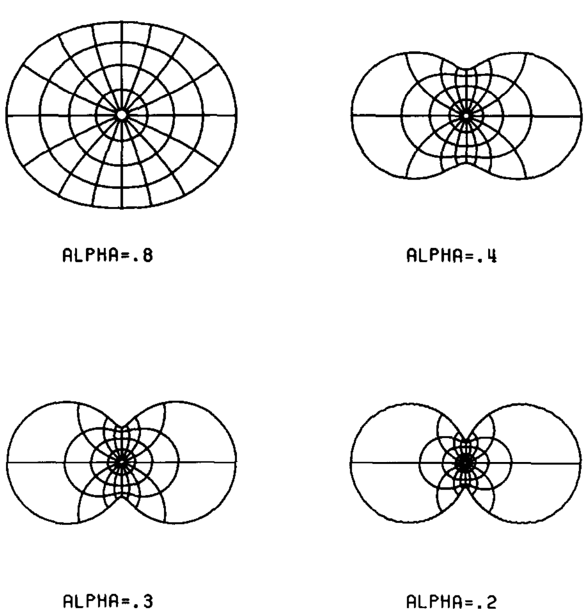

Fig. 3. Inverted ellipses with Wegmann's method.

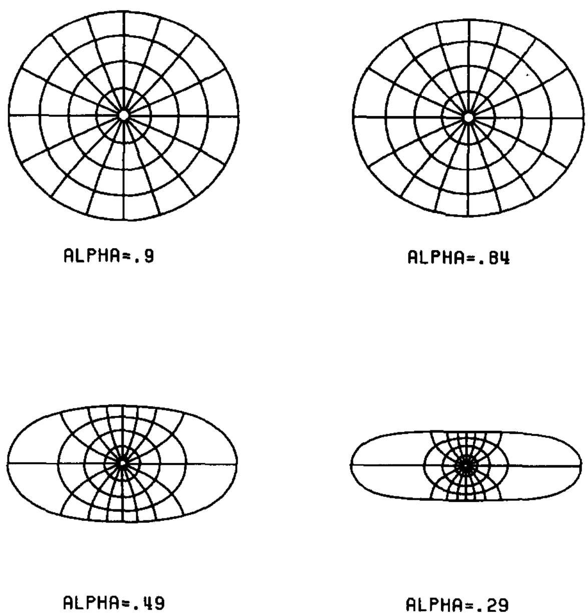

Fig. 4. Arctanh regions with Wegmann's method.

The examples below show, as Dubiner claims, that the stretching in g is not as severe as the crowding. It seems to be the stretching or loss of resolution that affects the accuracy of methods for g also, though it is more difficult to give accuracy estimates in terms of the number of mesh points N. This is indicated in, for instance, [23], where it may be noted that, especially for the Cassini oval and inverted ellipse, max $|g'|$ gives a good fit to the discretization error. Our α replaces Trummer's $((1-\alpha^2)/(1+\alpha^2))^{1/2}$ for Cassini ovals where

$$f(z)=\alpha z\left(\frac{2}{1+\alpha^2-(1-\alpha^2)z^2}\right) \quad \text{and} \quad \max |f'| = |f'(\pm 1)| = \mathrm{O}(\alpha^{-2}).$$

Also note for $g=f^{-1}$:

$$\max |g'| = \mathrm{O}(\alpha^{-1}).$$

The severe crowding apparently does not necessarily affect the methods for g as results from O'Donnell and Rokhlin's fast implementation of the Kerzman–Trummer method [17] and Hough's implementation of Symm's method [12] show.

Example 5.1 (*Inverted ellipse*, Fig. 3). $\partial\Omega$ is given by $\gamma(\sigma)=\rho(\sigma)\mathrm{e}^{\mathrm{i}\sigma}$ where $\rho(\sigma)=(1-(1-\alpha^2)\sin^2\sigma)^{1/2}$ for $0\leqslant\sigma\leqslant 2\pi$ and $0<\alpha\leqslant 1$. This map is derived from the familiar Joukowski map to the exterior of an ellipse. We note the following:

$$f(z)=\frac{2\alpha z}{1+\alpha-(1-\alpha)z^2}, \quad f'(0)=\frac{2\alpha}{1+\alpha}=\mathrm{O}(\alpha),$$

$$\max |f'| = |f'(\pm 1)| = \alpha^{-1}, \qquad \min |f'| = |f'(\pm \mathrm{i})| = \alpha^2.$$

Also note for $g=f^{-1}$:

$$g'(0)=\mathrm{O}(\alpha^{-1}), \qquad \min |g'| = \mathrm{O}(\alpha), \qquad \max |g'| = \mathrm{O}(\alpha^{-2}).$$

Here the crowding min $|g'|$ is algebraic in α.

Example 5.2 (*Arctanh*, see Fig. 4). We use the map from the disk to the infinite strip to generate the bounded domain corresponding to the function f given as follows:

$$f(z)=\frac{\operatorname{arctanh}(rz)}{\operatorname{arctanh}(r)}=\log\left(\frac{1+rz}{1-rz}\right)\Big/\log\left(\frac{1+r}{1-r}\right), \quad 0<r<1,$$

$$\alpha \sim -\frac{\pi}{2\log(1-r)}, \quad r\sim 1, \quad f'(0)=\frac{4}{\pi}\alpha=\mathrm{O}(\alpha),$$

$$\max |f'| = |f'(\pm 1)| \sim \frac{2\alpha}{\pi}\mathrm{e}^{\pi/2\alpha}, \qquad \min |f'| = |f'(\pm\mathrm{i})| \sim \frac{2}{\pi}\alpha.$$

Also note for $g=f^{-1}$:

$$g'(0)=\mathrm{O}(\alpha^{-1}), \qquad \min |g'| \sim \frac{\pi}{2\alpha}\mathrm{e}^{-\pi/2\alpha}, \qquad \max |g'| \sim \tfrac{1}{2}\pi\alpha^{-1}.$$

Here the crowding min $|g'|$ is exponential in α.

In [26] Zemach discusses the crowding for the map f to the interior of an ellipse and finds similar exponential crowding in α,

$$\max |f'| \sim \frac{\alpha^2}{2\pi} e^{\pi^2/4\alpha}.$$

Example 5.3 (*Slit disk*). The function $w = f(z)$, $f(0) = 0$, defined implicitly by the equation

$$\frac{w}{(1+w)^2} = \frac{\mu z}{(1+z)^2}, \quad 0 < \mu < 1,$$

maps the unit disk conformally onto the slit unit disk. The slit is the interval $\alpha \leqslant x \leqslant 1$ on the positive real axis with

$$\mu = \frac{4\alpha}{(1+\alpha)^2}, \quad \alpha = \frac{1-\sqrt{1-\mu}}{1+\sqrt{1-\mu}}.$$

For the points $e^{\pm i\theta}$, $0 < \theta = \theta(\alpha) < \pi$, that map to 1,

$$\sin\left(\tfrac{1}{2}\theta(\alpha)\right) = \frac{1-\alpha}{1+\alpha},$$

and consequently

$$\sin \pi\omega = \frac{4\sqrt{\alpha}\,(1-\alpha)}{(1+\alpha)^2},$$

where $\omega = \omega(\alpha) = (\pi - \theta(\alpha))/\pi$ is the harmonic measure of E, the arc on $|z| = 1$ that maps onto the full unit circle, $w = e^{it}$, $0 < t < 2\pi$. Hence the crowding is algebraic.

Example 5.4 (*Slit square*). We illustrate (5) by computing the Schwarz–Christoffel map from the disk to the unit square slit from i to iα using SCPACK [22]. E is then the portion of the boundary from i counterclockwise around the square back to i not including the slit. Table 1 lists $\omega(0, E, \Omega)$ for various values of α. We see there, roughly, the algebraic crowding

$$\omega(0, E, \Omega) \approx 1.2\sqrt{\alpha}.$$

Example 5.5 (*Rectangle*). Here we illustrate the estimates in Theorem 3.1 by computing the Schwarz–Christoffel map from the disk to the rectangle Ω with corners $(1, \frac{1}{2}w)$, $(-1, \frac{1}{2}w)$, $(-1, -\frac{1}{2}w)$, $(1, -\frac{1}{2}w)$ with SCPACK [22]. Let E be the side from $(1, -\frac{1}{2}w)$ to $(1, \frac{1}{2}w)$. Note that for this region the length is $l = 2$, the width is w and the area is $A = 2w$. The origin of the unit disk is mapped to $z_0 = (1 - l', 0)$, so $d(z_0, E, \Omega) = l'$. In Table 2, $\omega(z_0, E, \Omega)$ is given for various values of l' and $\frac{1}{2}w$, the aspect ratio of the rectangle. These values, computed with SCPACK, are in agreement with the exact values,

$$\omega(z_0, E, \Omega) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin\left(\frac{1}{2}n\pi\right)}{n} \frac{\sinh(n\pi(2-l')/w)}{\sinh(2\pi n/w)},$$

obtained by separation of variables. Since $\omega(z_0, E, \Omega) \sim 4e^{-\pi l'/w}/\pi$ for l' fixed and $w \downarrow 0$, $4e^{-\pi l'/w}/\pi$ is found to give a good estimate of $\omega(z_0, E, \Omega)$.

Table 1
Data for Example 5.4

α	$\omega(0, E, \Omega)$	$l/\sqrt{\alpha}$
$\frac{1}{2}$	0.762	1.08
$\frac{1}{4}$	0.572	1.14
$\frac{1}{8}$	0.418	1.18
$\frac{1}{16}$	0.301	1.20
$\frac{1}{32}$	0.215	1.21
$\frac{1}{64}$	0.153	1.22
$\frac{1}{128}$	0.108	1.22

Table 2
Data for Example 5.5

$\frac{1}{2}w$	l'	$\omega(z_0, E, \Omega)$	$\frac{4}{\pi}e^{-\pi l'/w}$
1.0	1.0	0.25	0.27
1.0	1.2	0.18	0.19
1.0	1.4	0.12	0.14
1.0	1.8	0.035	0.075
0.8	1.0	0.17	0.18
0.8	1.2	0.12	0.12
0.8	1.4	0.074	0.081
0.8	1.8	0.020	0.038
0.4	1.0	0.025	0.025
0.4	1.2	0.011	0.011
0.4	1.4	0.0052	0.0052
0.4	1.8	0.00086	0.0011
0.2	1.0	0.00049	0.00050
0.2	1.2	0.00010	0.00010
0.2	1.4	0.000021	0.000022
0.2	1.6	0.0000044	0.0000044
0.1	1.0	0.00000019	0.00000019

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