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On some relations among numerical conformal mapping methods

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Abstract: Gaier and Gutknecht have shown that many numerical methods for producing the conformal map from the unit disk to a simply connected region share a common theoretical basis as solutions of nonlinear integral equations arising from the Hilbert transform of a function of the boundary correspondence. We give a brief presentation of this classification and extend it somewhat to include some equations for the inverse correspondence, such as those of Menikoff and Zemach, Noble, and Schwarz–Christoffel. The use of explicit maps and the method of Bisshopp are also brought into this framework. An example illustrating the use of explicit maps is given.

Keywords: Numerical conformal mapping, conjugate function, Menikoff–Zemach equation, Schwarz–Christoffel transformation, integral equations.

1. Introduction

The so-called auxiliary functions and their conjugate relations were used in Gaier [12] to derive methods for approximating the conformal map f from the unit disk D to the interior Δ of a Jordan curve Γ: $\gamma(\eta)$. (Below the parameter of the curve η will generally be taken as θ, polar angle or σ, arclength. Notation is set in Fig. 1.) Recently Gutknecht [15,17] has specified this scheme more completely in terms of operators on function spaces. The use of this framework generally gives a nonlinear integral equation, involving the conjugation operator, for the boundary correspondence function, say $\sigma(t)$, or its inverse, $t(\sigma)$, where $f(e^{it}) = \gamma(\sigma(t))$. This equation is then solved by some iterative technique, for instance, a direct functional iteration with relaxation or a Newton method. The main computational cost here is the repeated application of the conjugation operator K using FFT's. The purpose of this paper is to provide a brief introduction to this framework, and relate it to certain other methods which do not necessarily compute the Fourier series. In the remainder of this section we present the relevant facts concerning the operator K and the map f and its derivative. Section 2 discusses the classical Theodorsen equation and the related equation of Menikoff and Zemach, which was also discussed by Gutknecht [17]. In Section 3 we derive the less well-known and related equations of Timman, Friberg, and Noble. Section 4 exploits a form of the auxiliary function suggested by

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Table 1
List of auxiliary functions

Auxiliary function	Related methods
(H1) $h(z) := f(z)$	Chakravarthy and Anderson, Fornberg, Wegmann
(H2) $h(z) := f(z)/z$	(Fornberg, Wegmann), Melentiev and Kulisch
(H3) $h(z) := \log f(z)/z$	Theodorsen, Menikoff–Zemach
(H4) $h(z) := \log f'(z)$	Timman, Noble, Dubiner(?)
(H5) $h(z) := \log z^2 f'(z)/f^2(z)$	Gutknecht
(H6) $h(z) := \log zf'(z)/f(z)$	Friberg
(H7) $h(z) := \log f'(z)/g(z)$	Ives, SC, Davis
(H8) $h(z) := g \circ f(z)$	composite methods
(H9) $h(z) := f \circ g(z)$	composite methods

Note that (H5) and (H6) are linear combinations of (H3) and (H4). Therefore, applying integration by parts to (H5) or (H6) would lead to linear combinations of the Menikoff–Zemach and Noble equations.

Ives to relate the equations of Noble and Davis and the Schwarz–Christoffel (SC) transformation. The use of singularities is also briefly discussed. Finally, in Section 5, the use of explicit maps is brought into this framework and illustrated in an example. A method due to Bisshopp is also discussed. Most of our derivations have been given elsewhere, but we believe that collecting them here will facilitate the comparison of the methods and indicate some new directions of investigation. A list of the standard auxiliary functions and the related methods is given in Table 1. Methods which relate perturbations of the map to perturbations of the boundary, as by Dubiner [9], Meiron et al. [35], and Menikoff and Zemach [36, section VI], are not considered here.

Suppose $\hat{h}_k$, $k \in Z$, are the Fourier coefficients of $h \in L^2(T)$, where T denotes the quotient space $R/2\pi Z$. Then

$$h = \sum_{k \in Z} \hat{h}_k \, e^{ikt}.$$

The *conjugation operator* K: $L^2(T) \to L^2(T)$ is then given in terms of the Fourier series

$$K(h(t)) = -i \sum_{k \in Z} \operatorname{sgn}(k) \hat{h}_k \, e^{ikt},$$

where $\operatorname{sgn}(k) = 1$ if $k > 0$, 0 if $k = 0$, and -1 if $k < 0$. We are also interested in its representation as a singular integral operator. If $h \in L^1(T)$, then for almost every $t \in T$

$$K(h(t)) = \frac{1}{2\pi} \mathrm{PV} \int_T \cot\left(\frac{t - \tilde{t}}{2}\right) h(\tilde{t}) \, d\tilde{t}. \tag{1.1}$$

Gutknecht uses the following fundamental theorem and its converse:

Theorem 1. *If $h \in H^1(D)$, then*
(a) $\operatorname{Im} h(e^{it}) - \operatorname{Im} h(0) = K \operatorname{Re} h(e^{it})$,
(b) $\operatorname{Re} h(e^{it}) - \operatorname{Re} h(0) = -K \operatorname{Im} h(e^{it})$, *and if $h \in H^1(D^c)$ then*
(a_e) $\operatorname{Im} h(e^{it}) - \operatorname{Im} h(\infty) = -K \operatorname{Re} h(e^{it})$,
(b_e) $\operatorname{Re} h(e^{it}) - \operatorname{Re} h(\infty) = K \operatorname{Im} h(e^{it})$.

He then regards the auxiliary function h as an image of f and its derivatives under an operator H on appropriate spaces. In the standard cases listed below, H and H^{-1} are given by simple formulas, e.g.

$$h(z) = Hf(z) = \log\frac{f(z)}{z} \quad \text{and} \quad f(z) = H^{-1}h(z) = z\,\mathrm{e}^{h(z)}.$$

Thus to derive an integral equation we select H and a conjugation relation (a) or (b). The choice of H should also assist in satisfying the normalization conditions. By integrating the integral form of K by parts, additional integral equations for the inverse boundary map, such as those of Menikoff–Zemach and Noble, may be derived. Similar unified approaches to deriving integral equations for conformal mapping have been suggested elsewhere, often using the Green's function. See, for instance, [37,38,60]. Henrici [22,23] also gives a concise treatment of the standard linear integral equations for the inverse boundary correspondence.

We will use the following results:

Theorem 2. *Let $\eta \in L^1\ (T)$ be of bounded variation and suppose that $\eta(t+\delta) - \eta(t) = 0(1/\log\delta)$ a.e. Then $K(\eta(t))$ may be represented by a Riemann–Stieltjes integral a.e.:*

$$K(\eta(t)) = \frac{1}{\pi}\int_T \log\left|\sin\frac{t-\tilde{t}}{2}\right| \mathrm{d}\eta(\tilde{t}) \tag{1.2}$$

Proof. Note that

$$-2\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\log\sin\frac{\tilde{t}-t}{2} = \cot\frac{t-\tilde{t}}{2} \quad \text{for } 0 < \frac{\tilde{t}-t}{2} < \pi.$$

Integrating (1.1) by parts and using our assumption, we have

$$\begin{aligned} K(\eta(t)) &= \frac{1}{\pi}\lim_{\delta\downarrow 0}\left[(\eta(t+\delta) - \eta(t-\delta))\log\sin\tfrac{1}{2}\delta + \int_{t+\delta}^{2\pi+t-\delta}\log\left|\sin\frac{t-\tilde{t}}{2}\right|\mathrm{d}\eta(\tilde{t})\right] \\ &= \frac{1}{\pi}\int_T \log\left|\sin\frac{t-\tilde{t}}{2}\right|\mathrm{d}\eta(\tilde{t}). \qquad \square \end{aligned}$$

In [51] von Wolfersdorf discusses the solution of equations involving this operator in terms of Riemann–Hilbert problems. In light of Wegmann's method, this may be of interest for numerical methods. It is convenient to apply the standard results for change of variables and reduction to a Riemann integral:

Suppose that t: $T_L \to T$ is onto and strictly increasing as a function from $[0, L]$ onto $[0, 2\pi]$, and $\eta \in L^1(T) \cap \mathrm{BV}(T)$. Then

$$\int_T \log\left|\sin\frac{t-\tilde{t}}{2}\right|\mathrm{d}\eta(\tilde{t}) = \int_{T_L}\log\left|\sin\frac{t(l)-t(\tilde{l})}{2}\right|\mathrm{d}\eta(t(\tilde{l})). \tag{1.3}$$

Note: If t were strictly decreasing the sign would change.

Next suppose $\eta' \in L^1(T_L)$. Then

$$\int_{T_L}\log\left|\sin\frac{t(l)-t(\tilde{l})}{2}\right|\mathrm{d}\eta(\tilde{l}) = \int_{T_L}\log\left|\sin\frac{t(l)-t(\tilde{l})}{2}\right|\eta'(\tilde{l})\,\mathrm{d}\tilde{l}. \tag{1.4}$$

We will also make use of the following facts. Here σ, θ, λ, ρ, and γ are as in Fig. 1 for appropriate Γ:

Fact 1. $(\mathrm{d}\sigma/\mathrm{d}t)/\rho = (\mathrm{d}\theta/\mathrm{d}t)/\cos\beta$.

Proof. Since σ is the arclength along the curve Γ: $\gamma(\theta) := \rho(\theta)\mathrm{e}^{\mathrm{i}\theta}$, starlike w.r.t. 0,

$$\mathrm{e}^{\mathrm{i}\lambda(\theta)} = \left(\frac{\mathrm{d}\rho}{\mathrm{d}\theta} + \mathrm{i}\rho\right)\mathrm{e}^{\mathrm{i}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}\sigma}.$$

Therefore

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}\theta} + i\rho\right)\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\sigma} = \mathrm{i}\cos\beta - \sin\beta. \qquad \square$$

Note that we see here that the ϵ-condition for the Theodorsen method, $|\rho'|/|\rho| < \epsilon < 1$, and $\beta = \arg(\rho' + \mathrm{i}\rho) - \frac{1}{2}\pi$ implies $|\beta| < \frac{1}{4}\pi$.

Fact 2. $\arg f'(\mathrm{e}^{\mathrm{i}t}) = \beta(t) + \theta(t) - t = \lambda(t) - t - \frac{1}{2}\pi$.

Fact 3. $|f'(\mathrm{e}^{\mathrm{i}t})| = \mathrm{d}\sigma/\mathrm{d}t$.

Fact 4. *The normalization of the tangent angle* λ *is given by* $\int_0^{2\pi}\lambda(t)\,\mathrm{d}t = 2\pi \arg f'(0) + 3\pi^2$.

Proof. Use the fact that $\arg f'(z) = \operatorname{Im}\log f'(z)$ is harmonic in $|z| < 1$. $\square$

Fact 5. $(1/2\pi)\mathrm{PV}\int_{\mathrm{T}}\cot((t-\tilde{t})/2)\tilde{t}\,\mathrm{d}\tilde{t} = -2\log 2$.

Proof. See, for instance, Ahlfors [1, p. 170, problem 5].

Denoting the approximation to any function g by $\tilde{g}$, and setting $\|g\| = \|t(\mathrm{e}^{\mathrm{i}t})\|\infty$ below, we give some rough estimates of the error $\|f-\tilde{f}\|$. The accuracy estimates may also be stated in terms of the number N_{F} of Fourier coefficients needed to achieve a certain minimum order-of-magnitude level of accuracy. Zemach [58,59], has shown that $N_{\mathrm{F}} \gtrsim \frac{1}{2}\|f'\|$. Our estimates will express $\|f'\|$ in terms of $\|h'\|$ or $\|f-\tilde{f}\|$ in terms of $\|h-\tilde{h}\|$, according to convenience. The point of these estimates is to indicate how the form of the auxiliary function might influence the accuracy of the solution. Two main features of Γ affect this accuracy, namely, its local smoothness and, more dramatically, the global 'thinness' of Δ. For the interior problem for a thin region $\|f'\|$ may be very large, making the problem ill-conditioned. The accuracy estimates in Section 3, for instance, though probably crude, show how $\|f'\|$ might influence the method.

In the case where $\tilde{f}$ is given by the Taylor series of f, truncated after N terms, and the boundary curve is analytic, the effect of $\|f'\|$ can be seen more explicitly. Let R be the modulous of the singularity of f nearest the unit disk. Then error estimates of the form $\|f-\tilde{f}\| = \mathrm{O}(R^{-N/2})$ fit the numerical results for a wide range of N and R; see [54]. Consider the three popular test cases: the families of maps to the interior of the circle, the inverted ellipse, and the Cassini oval. Normalize the curves to have diameter 2, and let the thinness α be the distance of $f(0)$ from the boundary. Then $R = 1 + \delta(\alpha)$ and $\|f'\| = \mathrm{O}(1/\delta(\alpha))$, where $\delta(\alpha) =$

$O(\alpha)$, $O(\alpha)$, and $O(2\alpha^2)$, respectively, for the three cases. Thus $\| f-\tilde{f}\| = C\exp(-N/(2\|f'\|))$. The effect of $\|f'\|$ and Zemach's rule are both seen explicitly here. (Actually, Cauchy estimates seem to give a dependence of C on $\|f'\|$, too.)

We now classify various methods according to their auxiliary functions, give some derivations using integration by parts and comment on the standard numerical procedures. Solutions of the resulting nonlinear equations are generally obtained by direct or Newton iterations. The equations for the interior problem will generally be given, since the exterior problem just changes the sign of K. However the normalization may also have to be treated differently. Again we refer to Gutknecht for further details. Also Γ, with its mapping function, f, and various explicit functions, g, will be normalized so that $\|g\| = \|f\| = 1$.

(H1) $h(z) := f(z)$.

Accuracy: $\|f-\tilde{f}\| = \|h-\tilde{h}\|$.

The method of Chakravarthy and Anderson [7], which discretizes the Cauchy–Rieman equations, and the Newton methods of Wegmann [53,54] and Fornberg [10] are included in this family. Gutknecht handles the normalization of f for Wegmann's method more easily by including it under (H2).

(H2) $h(z) := f(z)/z$.

Accuracy: $\|f-\tilde{f}\| = \|h-\tilde{h}\|$.

If (a) is selected in Theorem 1, we arrive at the method of Melentiev and Kulisch [34] which attempts to solve

$$\theta(t) - t = \arctan\frac{K[\rho(\theta(t))\cos(\theta(t)-t)]}{\rho(\theta(t))\cos(\theta(t)-t)}$$

by direct iteration.

2. The equations of Theodorsen and Menikoff–Zemach

(H3) $h(z) := \log\dfrac{f(z)}{z}$ and $h(e^{it}) = \log\rho(\theta(t)) + i(\theta(t)-t)$, Γ starlike w.r.t. 0.

Accuracy: $\|f-\tilde{f}\| = \|e^h\|\,\|1-e^{\tilde{h}-h}\| = \|f\|O(\|h-\tilde{h}\|)$.

(a) gives $\theta(t) - t = K[\log\rho(\theta(t))]$. This is the Theodorsen integral equation [43] for the interior problem. It can be solved by various direct iteration methods, as in Gutknecht [14,16]. It can also be solved by Newton-like methods, as in Gaier [12] and, via certain Riemann–Hilbert problems, see Hübner [26]. See also Vertgeim [50].

By applying integration by parts to K, we arrive at the following equation for the *inverse* boundary correspondence for the interior problem:

$$\theta - t(\theta) = \frac{1}{\pi}\int_T \log\left|\sin\frac{t-\tilde{t}}{2}\right| \mathrm{d}\log\rho(\theta(\tilde{t})) = \frac{1}{\pi}\int_T \log\left|\sin\frac{t(\theta)-t(\tilde{\theta})}{2}\right| \frac{\rho'(\tilde{\theta})}{\rho(\tilde{\theta})}\,\mathrm{d}\tilde{\theta}. \quad (2.1)$$

The first equality holds since $\rho \neq 0$, while the second follows from (1.3), $\theta(t(\tilde{\theta})) = \tilde{\theta}$, and (1.4) when $\rho' \in L^1(T)$. We do not know if this method has ever been tried.

(b) gives $\log\rho(\theta(t)) - \log|f'(0)| = -K[\theta(t)-t]$.

Menikoff and Zemach [36] consider this equation and Theodorsen's equation in various geometries. However their main contribution involves applying integration by parts to (b). Using (1.2) and Fact 5 with

$$-2\log 2 = \frac{1}{\pi}\int_T \log\left|\sin\frac{\theta-\tilde{\theta}}{2}\right| \mathrm{d}\tilde{\theta}$$

to remove the logarithmic singularity for $t(\theta)$, we get

$$\log\rho(\theta) - \log|f'(0)| = -\frac{1}{\pi}\int_T \log\left|\frac{\sin\frac{1}{2}(t(\theta)-t(\tilde{\theta}))}{\sin\frac{1}{2}(\theta-\tilde{\theta})}\right| \mathrm{d}\tilde{\theta}. \tag{2.2}$$

They solve a discrete version of this equation by Newton's method in $O(N^3)$ operations. For thin regions, crowding of mesh points presents less of a problem for $t(\theta)$ than spreading does for $\theta(t)$. Thus fewer points are needed in the Γ-plane to represent $t(\theta)$ accurately. Equation (2.1) would presumably have the same advantage. The use of the FFT does not seem to be possible in either case.

3. The equations of Timman, Friberg and Noble

(H4) $h(z) := \log f'(z)$ and $h(e^{it}) = \log\dfrac{\mathrm{d}\sigma}{\mathrm{d}t} + i(\lambda(\sigma(t)) - t - \frac{1}{2}\pi)$ by Facts 2 and 3.

Accuracy: Here we have $f(z) = f(1) + \int_1^z e^{h(w)}\mathrm{d}w$, where we integrate along an arbitrary path in the unit disk, say the straight line from 1 to z. Then

$$|f-\tilde{f}| \leqslant \left(\int_1^z |e^h|\mathrm{d}\sigma\right)\|1-e^{\tilde{h}-h}\|.$$

So

$$\|f-\tilde{f}\| = \|f'\| O(\|h-\tilde{h}\|).$$

Since also $\|h-\tilde{h}\| = \|\log f' - \log \tilde{f}'\|$, the largest absolute errors are likely to occur where f' is the largest or smallest. The latter case may occur where there are zeros of f' near the disk, i.e. where the conformality of the map breaks down. Wegmann [54] reports some numerical evidence of loss of accuracy in this latter case for his method which, however, uses $h := f$.

(a) $\lambda(\sigma(t)) - t - \frac{1}{2}\pi = K[\log(\mathrm{d}\sigma/\mathrm{d}t)]$. This might make sense for a convex Γ parametrized by λ.

(b) $\log(\mathrm{d}\sigma/\mathrm{d}t) - \log|f'(0)| = -K[\lambda(\sigma(t)) - t - \frac{1}{2}\pi]$. This is the analog for the interior problem of the equation of Timman [23,46]. See also James [29] and Birkhoff, Young and Zarantello [5]. This equation is not so useful, computationally, since there is no general way to impose the normalization condition, $f(0) = 0$. The exterior problem can be handled, though, and the interior problem can always be treated as an exterior problem with two inversions of the plane. Thus Gutknecht suggests the following auxiliary function, which is a combination of (H3) and (H4):

(H5) $$h(z) = \log\frac{z^2 f'(z)}{f^2(z)} = \log f'(z) - 2\log\frac{f(z)}{z}.$$

Accuracy: Since

$$f(z)=\left[\frac{1}{f(1)}-\int_1^z \frac{e^{h(w)}}{w^2}\,dw\right]^{-1}$$

we have

$$|f-\tilde{f}| \leqslant \left| f\tilde{f}\int_1^z |e^h/w^2|\,d\sigma \right| \|1-e^{h-\tilde{h}}\| \leqslant |f\tilde{f}|\ \|f'/f^2\|\ O(\|h-\tilde{h}\|)$$

Therefore $\|f-\tilde{f}\| = O(\|f'\|\ \|h-\tilde{h}\|/\alpha^2)$, where $\alpha = \|f\|/\min|f(z)|$ for $|z|=1$.

(H5) is similar to the function which gives Friberg's method:

$$\text{(H6)}\quad h(z) := \log\frac{zf'(z)}{f(z)} = \log f'(z) - \log\frac{f(z)}{z} \quad\text{and}\quad h(e^{it}) = \log\frac{d\theta/dt}{\cos\beta} + i\beta.$$

Accuracy: Here

$$f(z)=f(1)z\exp\left(\int_1^z \frac{e^{h(w)}-1}{w}\,dw\right).$$

Therefore, with $\tilde{f}(1)=f(1)$,

$$|f-\tilde{f}| \leqslant O(|f|\ \|h-\tilde{h}\|\ 2\|f'/f\|)$$

Therefore $\|f-\tilde{f}\| = O(\|f'\|\ \|h-\tilde{h}\|/\alpha)$, with α as defined above.

(a) gives

$$\beta(\theta(t)) = K\left[\log\left(\frac{d\theta(t)/dt}{\cos\beta(\theta(t))}\right)\right].$$

This equation does not appear to be solvable by direct iteration, since it does not seem possible to parametrize Γ globally by β.

(b) gives

$$\log(d\theta/dt) = \log(\cos\beta(\theta(t))) - K[\beta(\theta(t))].$$

This is Friberg's equation [11]. The equations of Timman and Friberg are solved by direct iteration. For remarks on convergence see Gutknecht. The results of Friberg's analysis are given in Warschawski [52]. The conditions for (linear) convergence are that $|\beta|$, $|\beta'|$ and $|\beta''|$ are all $\leqslant \epsilon < \frac{1}{4}\pi$. The first of these is just the ϵ-condition for the Theodorsen equation. As we note below, Friberg's equation is the sum of Timman and (H3b). Experiments by Halsey [19] indicate that Timman's method converges much better than Theodorsen's for thin regions. This suggests that an ϵ-condition on β might not be necessary, but leaves open the need for an ϵ-condition on β' and possible poor performance of Timman's method for a bumpy near-circle, where β' is large. It might be interesting to look at such a near-circle with $|\beta| < \epsilon < \frac{1}{4}\pi$, but $|\beta'| > \frac{1}{4}\pi$ where Theodorsen's method would succeed but Timman's method might fail. Kaiser [30] has done further analysis of convergence for this case.

Moreover, the asymptotic analysis of Dubiner [9] and Zemach [58] shows that whereas $d\sigma/dt$ (or $d\theta/dt$) may be very large for thin regions, $\log d\sigma/dt$ remains well-behaved. The good convergence properties of the discrete Timman equation will not overcome the need for large N

to represent thin regions accurately, i.e., Zemach's $N_F \gtrsim \frac{1}{2}\mathrm{d}\sigma/\mathrm{d}t$ rule remains true. See also Meiron, Israeli and Orszag [35].

Since $\sigma(t)$ in the method of Timman and $\theta(t)$ in Friberg's are updated in each step by integrating $\mathrm{d}\sigma/\mathrm{d}t$ and $\mathrm{d}\theta/\mathrm{d}t > 0$, respectively, the disordering of points which may affect other methods such as Theodorsen, Fornberg, or Wegmann is avoided; see Halsey [19]. Newton iterations based on solving Riemann–Hilbert problems may also be possible here. Timman's method may also be useful for providing a good initial guess for quadratically convergent methods such as Wegmann's or Fornberg's when the region is thin, thus avoiding additional coding for explicit maps or continuation. The main subroutines required by the FFT methods are the FFT routine and a subroutine for K.

As we noted, since K is linear, Timman's equation,

$$\log(\mathrm{d}\sigma/\mathrm{d}t) - \log|f'(0)| = -K(\lambda - t - \tfrac{1}{2}\pi)$$

combined with (H3b),

$$-\log \rho + \log|f'(0)| = K(\theta - t)$$

gives

$$\log(\mathrm{d}\sigma/\mathrm{d}t) - \log \rho = -K(\beta),$$

and Fact 1 gives Friberg's equation,

$$\log(\mathrm{d}\theta/\mathrm{d}t) - \log(\cos\beta) = -K(\beta).$$

Taking such linear combinations of the integral equations is clearly equivalent to taking linear combinations of their auxiliary functions and then applying Theorem 1. We may then apply integration by parts to K. Here, the two cases of interest are the Menikoff–Zemach equation, using (H3), and the Noble equation (3.1), using (H4). Applying integration by parts with (H5) or (H6) will just result in linear combinations of these equations; see Table 1. The Noble equation appears in various forms in Noble [39], Andersen et al. [2], and Woods [57]:

$$\log\frac{\mathrm{d}\sigma}{\mathrm{d}t} - \log|f'(0)| = -2\log 2 - \frac{1}{\pi}\int_T \log\left|\sin\frac{t-\tilde{t}}{2}\right| \mathrm{d}\lambda(\tilde{t}). \tag{3.1}$$

If $\mathrm{d}\lambda/\mathrm{d}t \in L^1(T)$, we may use (1.3) and (1.4) and $\mathrm{d}\lambda(\sigma)/\mathrm{d}\sigma = \kappa(\sigma)$, the curvature of the curve Γ of length L, to rewrite (3.1) as

$$\log\frac{\mathrm{d}t(\sigma)}{\mathrm{d}\sigma} + \log|f'(0)| = 2\log 2 + \frac{1}{\pi}\int_{T_L} \log\left|\sin\frac{t(\sigma)-t(\tilde{\sigma})}{2}\right| \kappa(\tilde{\sigma})\,\mathrm{d}\tilde{\sigma}. \tag{3.2}$$

If λ is strictly increasing, we get

$$\begin{aligned}\log\frac{\mathrm{d}t}{\mathrm{d}\sigma} + \log|f'(0)| &= \frac{1}{\pi}\int_T \log\left|\frac{\sin\frac{1}{2}(t(\lambda)-t(\tilde{\lambda}))}{\sin\frac{1}{2}(\lambda-\tilde{\lambda})}\right| \mathrm{d}\tilde{\lambda}\\ &= \frac{1}{\pi}\int_{T_L} \log\left|\frac{\sin\frac{1}{2}(t(\sigma)-t(\tilde{\sigma}))}{\sin\frac{1}{2}(\lambda(\sigma)-\lambda(\tilde{\sigma}))}\right| \kappa(\tilde{\sigma})\,\mathrm{d}\tilde{\sigma}.\end{aligned} \tag{3.3}$$

If a parametrization of Γ by λ is known, discretizing the first line of (3.3) may advantageously distribute more mesh points along sections of Γ of greatest curvature. We may ask whether this might be more accurate than the second variant in (3.3), where mesh points would be distributed evenly in σ. For a related point of view see Hoidn [25], where a reparametrization is used to treat corners with the Symm's equation. Dubiner [9] also alludes to (3.3). For an application of this equation see [35, p. 354, after Eq. 3.4].

The problem of satisfying the normalization condition $f(0)=0$, which effects the interior version of the Timman equation, would seem to occur here too. However (3.1) is of independent interest, since we may use it to derive a form of the Schwarz–Christoffel transformation. Suppose Γ is a polygon with n corners at $\sigma_i = \sigma(t_i)$ with interior angles $\pi - \Delta\lambda_i$, $i = 1, \ldots, n$. Then for $t \neq t_i$ and the change in the tangent angle $|\Delta\lambda_i| < \pi$ we have

$$\log\frac{\mathrm{d}\sigma}{\mathrm{d}t} - \log|f'(0)| + 2\log 2 = -\frac{1}{\pi}\int_T \log\left|\sin\frac{t-\tilde{t}}{2}\right|\mathrm{d}\lambda(\tilde{t})$$
$$= -\frac{1}{\pi}\sum_{i=1}^{n}\log\left|\sin\frac{t-t_i}{2}\right|\Delta\lambda_i.$$

This gives

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \tfrac{1}{4}|f'(0)|\prod_{i=1}^{n}\left|\sin\frac{t-t_i}{2}\right|^{-\Delta\lambda_i/\pi}. \tag{3.4}$$

Since $\Delta\sigma_i = \int_{t_i}^{t_{i+1}}(\mathrm{d}\sigma/\mathrm{d}t)\mathrm{d}t$ are the known lengths of the sides of Γ, $i = 1, \ldots, n$ and $t_{n+i} = t_i$, we have the following n equations for the $n+1$ parameters $|f'(0)|$, $t_1, \ldots, t_n$

$$\Delta\sigma_i = \tfrac{1}{4}|f'(0)|\int_{t_i}^{t_{i+1}}\left|\sin\frac{t-t_i}{2}\right|^{-\Delta\lambda_i/\pi}\mathrm{d}t, \quad i = 1, \ldots, n.$$

Note for $|\Delta\lambda_i| < \pi$ the singularity is integrable; see [2, p. 154]. Also see Koppenfels and Stallmann [33, p. 159] for a connection to Theodorsen's equation.

4. The Ives form

(H7) $h(z) := \log(f'(z)/g(z))$, where $g(z)$ is given explicitly.

Thus $f'(z) = g(z)\mathrm{e}^{h(z)}$, a form suggested by Ives in his interesting survey [28]. This case includes Schwarz–Christoffel (SC) and the continuous SC of Davis [8,42] when $g(z)$ is a product of SC factors.

Accuracy: $f(z) = f(0) + \int_0^z g(w)\mathrm{e}^{h(w)}\mathrm{d}w$, so

$$\|f-\tilde{f}\| = \left\|\int_0^z f'(w)\big(1-\exp(\tilde{h}(w)-h(w))\big)\,\mathrm{d}w\right\| \leqslant \mathrm{O}(\|f'\|\,\|\tilde{h}-h\|).$$

We now establish some relationships between the Schwarz–Christoffel transformation, the Davis equation, the Ives form, and the Noble equation.

SC ⇒ Davis. Let $\tilde{\Gamma}$ be a polygon with n corners on Γ. Again let $\Delta\tilde{\lambda}_i$ be the change in the tangent angle at the corners of $\tilde{\Gamma}$ which should include any corners of Γ. The SC map $\tilde{f}$ for $\tilde{\Gamma}$ maps the unknown $\tilde{z}_i$ to these corners $\tilde{f}(\tilde{z}_i)$, $i = 1, \ldots, n$ and satisfies, for some constant c,

$$\frac{\mathrm{d}\tilde{f}}{\mathrm{d}z} = C\exp\left(-\frac{1}{\pi}\sum_{i=1}^{n}\log(z-\tilde{z}_i)\,\Delta\tilde{\lambda}_i\right).$$

Taking the limit as $n \to \infty$, $\tilde{\Gamma} \to \Gamma$ and $\max|\tilde{z}_i - \tilde{z}_{i-1}| \to 0$ we obtain Davis' equation for f:

$$\frac{\mathrm{d}f}{\mathrm{d}z} = C \exp\left(-\frac{1}{\pi}\int_\Gamma \log(z-\tilde{z})\, \mathrm{d}\tilde{\lambda}\right). \tag{4.1}$$

The numerical problem is to determine the $\tilde{z}_i$ for the case of the polygon and $\lambda(z)$ for the case of the more general curve Γ. The Riemann–Stieltjes integral incorporates jumps in λ as the corners. Davis uses composite techniques, assuming λ to be quadratic, to evaluate the integral explicitly on the smooth sections.

Davis ⇒ Ives. Suppose Γ has m corners, z_i, $i = 1, \dots, m$. Davis shows how the product, $g(z)$, of the SC factors for the corners can be factored out of (4.1), to get

$$\frac{\mathrm{d}f}{\mathrm{d}z} = C(z-z_i)^{-\Delta\lambda_i/\pi} \exp\left(-\frac{1}{\pi}\sum_{i=1}^{m}\int_{z_i}^{z_{i+1}} \log(z-\tilde{z})\, \mathrm{d}\lambda(\tilde{z})\right)$$

where $z_{m+1} = z_1$. Thus f is of the form

$$\mathrm{d}f/\mathrm{d}z = g(z)\, \mathrm{e}^{h(z)}. \tag{4.2}$$

This is the so called 1-step form suggested by Ives as the method of choice of the U.S. aerodynamics community over composite methods (below) if the FFT is used to compute $h(z)$. Bauer et al. [3] use this form for the exterior map to an airfoil with a corner at the trailing edge. Other choices of $g(z)$ are given in Ives' survey.

Apparently general behavior may be resolved by using $g(z)$ to place singularities on or near appropriate sections of Γ. Fornberg (private communication) has suggested treating the mapping problem by distributing singularities around the unit disk. Papamichael and his coworkers, e.g. [40], have improved the accuracy of certain kernel methods for the inverse map by exact treatment of singularities. It might be expected that something similar can be done here, e.g. by choosing $h(z) := \log(f(z)/g(z))$. The $f(z) = g(z)\mathrm{e}^{h(z)}$. If, for example, one wishes to map to the inverted ellipse where the singularities $z_\pm$ are known, one could choose

$$g(z) = z(1 - z/z_+)^{-1}(1 - z/z_-)^{-1}.$$

In this case h would be constant. Even if this scheme worked in test cases, a method for approximating the dominant singularities of f would be needed for general analytic Γ, and some similar scheme would be needed for the practical case when Γ is, say, a cubic spline.

Davis ⇒ Noble (see [2]): Consider the Davis equation:

$$\log f'(\mathrm{e}^{\mathrm{i}t}) = \log C - \frac{1}{\pi}\int_T \log(\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{\mathrm{i}\tilde{t}})\, \mathrm{d}\lambda(\tilde{t}). \tag{4.3}$$

Using Fact 4 and the branch of log with $\arg x = -\pi$ for $x < 0$, a calculation gives

$$\begin{aligned}\int_T \log(\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{\mathrm{i}\tilde{t}})\, \mathrm{d}\lambda(\tilde{t}) = \int_T \log\left|\sin\frac{\tilde{t}-t}{2}\right| \mathrm{d}\lambda(\tilde{t}) - \mathrm{i}\pi\lambda(t) + \mathrm{i}\pi t + \pi 2\log 2 \\ + \mathrm{i}\pi 2\lambda(0) - \frac{\mathrm{i}\pi^2}{2} - \mathrm{i}\pi \operatorname{Arg} f'(0).\end{aligned} \tag{4.4}$$

We also have

$$\log C = \log f'(0) - \mathrm{i}2\pi + \mathrm{i}2\lambda(2\pi) - \mathrm{i}2 \operatorname{Arg} f'(0) - \mathrm{i}3\pi. \tag{4.5}$$

Since $\log f'(\mathrm{e}^{\mathrm{i}t}) = \log(\mathrm{d}\sigma/\mathrm{d}t) + \mathrm{i}(\lambda(t) - t - \frac{1}{2}\pi)$, we obtain Noble's equation (3.1) by combining (4.3), (4.4), and (4.5).

5. The use of explicit maps

We next suggest two forms of h encompassng the classical technique of composition with explicit maps, g, which may be computed exactly.

(H8) $h(z) := (g \circ f)(z)$.

Accuracy: $\| f - \tilde{f} \| = \| g^{-1} \circ h - g^{-1} \circ \tilde{h} \| \leqslant \| g^{-1\prime} \| \, \| h - \tilde{h} \|$

Thus the accuracy of $\tilde{f}$ will be indicated by $\| g^{-1\prime} \| \, \| h' \|$ is this case. Here g may be some composition of explicit maps such as Koebe, osculation, Karman–Trefftz, corner-removers, etc., chosen, e.g. by Grassmann's algorithm [13], to map Δ to a more nearly circular region. One could even imagine g^{-1} as a composition of very accurate Taylor series maps. For a thin region and fixed N the Taylor series map $\tilde{h}$ to the near-circle should be more accurate than the Taylor series map to the region. Unfortunately this extra accuracy is lost in amplification by $\| g^{-1\prime} \|$. However, the use of explicit maps can, in our experience, replace continuation. We intend to report some experiments composing the Grassmann maps with the Fourier series maps of Fornberg and Wegmann in a subsequent paper.

(H9) $h(z) := (f \circ g)(z)$.

Here $g: D \to D$ conformally, and is thus a fractional linear transformation

Accuracy: $\| f - \tilde{f} \| = \| h(g^{-1}) - \tilde{h}(g^{-1}) \| = \| h - \tilde{h} \|$.

The accuracy of $\tilde{f}$ here will be indicated just by $\| h' \|$, since g is exact.

The example of Fig. 1 illustrates the comparison of (H1), (H8) and (H9) with known explicit maps. We wish to find f mapping the unit disk to the interior of the inverted ellipse of thinness α, but with $f(-1+\beta) = 0$ and $f'(-1+\beta) > 0$ for small α and β. f is the known composite map $f(z) = g(h(z))$ where

$$h(z) = \frac{z + 1 - \beta}{1 + (1 - \beta) z},$$

the fractional linear transformation and

$$g(z) = \frac{2\alpha z}{(1+\alpha) - (1-\alpha) z^2},$$

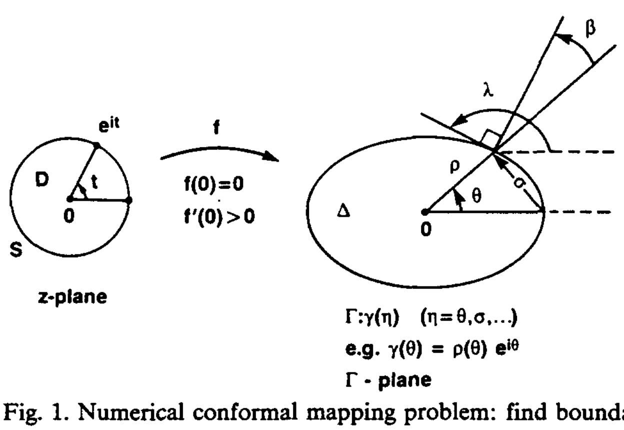

Fig. 1. Numerical conformal mapping problem: find boundary correspondence, e.g. $\sigma(t)$.

the well-known map to the interior of an ellipse of major axis $1/\alpha$ and minor axis 1, inverted in the unit circle. These functions have maximum derivatives at -1 of $O(1/\beta)$ and $O(1/\alpha)$, respectively.

Using (H1) we find that the accuracy of a Taylor series representation $\tilde{f}$ of f is given by $\| f' \| = f'(-1) = g'(-1)h'(-1) = O(1/(\alpha\beta))$.

Using (H8) the circle map is represented by its Taylor series $\tilde{h}$. Its accuracy is $\| h' \| = h'(-1) = O(1/\beta)$ and it is magnified by $\| g' \| = g'(-1) = O(1/\alpha)$. The accuracy of $\tilde{f} = g \circ \tilde{h}$ is thus $\| g' \| \| h' \| = O(1/(\alpha\beta))$, the same as (H1).

Using (H9) the exact map g is the circle map and $\tilde{h}$ is the Taylor series map for the inverted ellipse. The error in g will be negligible and the accuracy of $\tilde{h}$ will be $\| h' \| = O(1/\alpha)$. So the accuracy of $\tilde{f} = \tilde{h} \circ g$ is $O(1/\alpha)$, a clear improvement for small β.

There does not seem to be any way to exploit the third normalization condition unless it is not required and the circle can be rotated arbitrarily so that the maximum and minimum derivatives line up. Presumably the best strategy using the circle maps first would be to find the point w_0 in the target region farthest from the boundary and map the origin to it. w_0 is such that

$$\sup_{z\in\Gamma} | w_0 - z | = \inf_{w\in\Delta} \sup_{z\in\Gamma} | w - z |.$$

If the desired normalization is $f(z_0) = w_0$, $z_0 \neq 0$, we can map z_0 to 0 with a linear transformation. If we want $f(0) = w_1 \neq w_0$ we can find the map with $h(0) = w_0$ and then find $z_1 = h^{-1}(w_1)$ by applying Newton's method to $h(z_1) = w_1$. Finally map 0 to z_1 by a linear transformation.

It is not clear to this author whether the above idea can be implemented, and whether it would yield the smallest $\| f' \|$ in all cases. For near circles, Ives [27] computes w_0 as the centroid. To

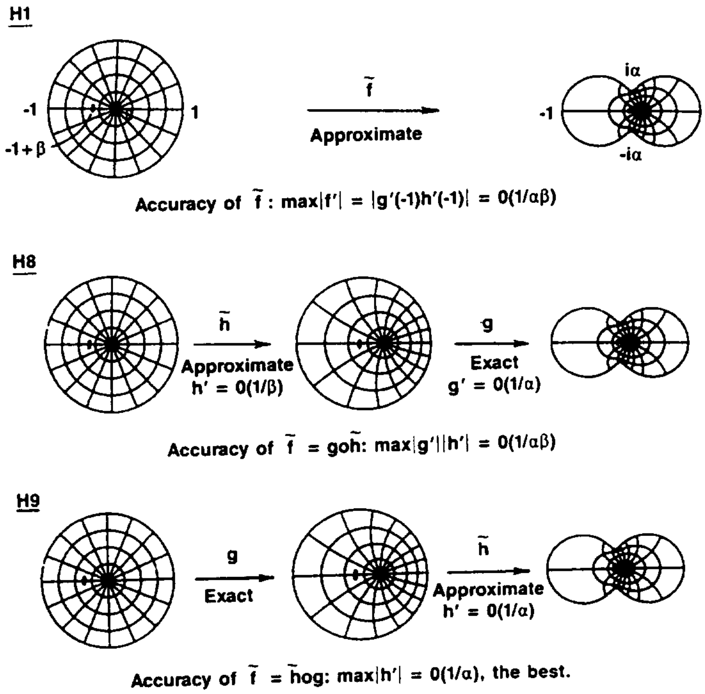

Fig. 2. Comparison on (H1), (H8) and (H9).

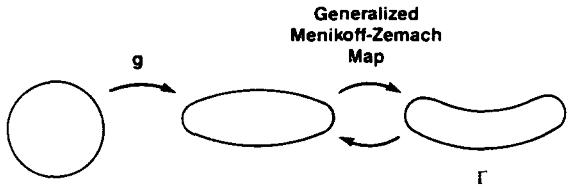

Fig. 3. Map to a thin region.

characterize the point w_0 for quite odd regions would probably be unnecessary, since such a case would most likely be beyond the range of Fourier series maps anyway. A method proposed by Bisshopp [6] seems to find this best map f. He solves a least squares problem using FFT's,

$$E^2 = \lim_{r \to 1} \int_0^{2\pi} |f(r\,e^{it}) - \gamma(\sigma(t))|^2 \, dt,$$

with

$$f(r\,e^{it}) = \sum_{k=0}^{\infty} a_k r^k e^{ikt}.$$

The conditions $\partial E / \partial a_k = 0$ give values of the a_k. The vanishing of the first variation of E^2 leads to a Newton method for $\sigma(t)$. Once f is found the desired normalization may be satisfied with circle maps. The normalization may also be imposed from the outset, but Bisshopp observes that this leads to loss of accuracy. Two questions suggest themselves. Is $f(0)$ for Bisshopp's map equal to w_0 above? Can Bisshopp's method be posed as a Riemann–Hilbert problem and solved in Wegmann's fashion?

The examples above indicate that it is best to use explicit maps first, since otherwise errors in the approximate map will just be amplified due to spreading by the explicit map. Menikoff and Zemach [37] give a generalization of their methods to maps between arbitrary regions. This suggests the following strategy for mapping from D to a thin region Γ. Use for g, for instance the known explicit map to the ellipse, or the inverted Grassmann maps for Γ, follow these by the generalized Menikoff–Zemach map between the ellipse as the image of the circle under the inverted Grassmann maps and the region Γ, as in Fig. 2. Severe crowding may then be avoided in the Menikoff–Zemach map. Another good strategy might be to start with canonical regions which avoid crowding. However, in this case fast methods may be lost.

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